Funkcije več spremenljivk.

$\mathbb{R}^n = \underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{n\text{-krat}}$, je vektorski prostor nad $\mathbb{R}$.

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

$x+y$, λx po komponentah.

Standardna baza: $e_j = (0, \ldots, 0, \overset{j\text{-to mesto}}{1}, 0, \ldots 0)$

Za $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ pišemo x, y, z namesto x_1, x_2. Bazni so $\vec{i}, \vec{j}, \vec{k}$

Skalarni produkt: $x \cdot y = x_1y_1 + \ldots + x_ny_n$

$\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$

Standardna metrika: $\|x-y\|_2 = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2}$

Odprta krogla s središčem v točki $a \in \mathbb{R}^n$ in polmerom $r>0$: $K(a,r) = \{x \in \mathbb{R}^n, d_2(x,a) < r\}$

Zaprta krogla: $\bar{K}(a,r) = \{x \in \mathbb{R}^n, \|x-a\| \le r\}$

Sfera: $S(a,r) := \{x \in \mathbb{R}^n, \|x-a\|_2 = r\}$

$(\mathbb{R}^n, d_2)$ je metrični prostor. Dobimo topologijo na $\mathbb{R}^n$: $\overset{\text{odprta}}{\mathcal{O}^{odp}}$, $\overset{\text{zaprta}}{\mathcal{Z}}$

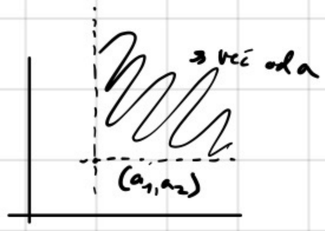
Kompaktna množica: za vsako pokritje obstaja končno podpokritje

Vedno so zaprte in omejene. V $\mathbb{R}^n$ velja tudi obrat, zato: Kompaktno $\Leftrightarrow$ omejeno in zaprto.

$a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$

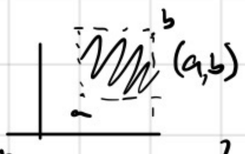
$a \le b \Leftrightarrow \forall j = 1, \ldots, n \;\; a_j \le b_j$

$a < b \Leftrightarrow \forall j = 1, \ldots, n \;\; a_j < b_j$



Naj bo $a<b$ in $a, b \in \mathbb{R}^n$.

Odprti kvader: $(a,b) := \{x \in \mathbb{R}^n, a < x < b\}$

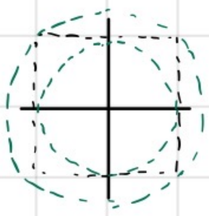


Č $a \le b$: Zaprti kvader: $[a,b] := \{x \in \mathbb{R}^n, a \le x \le b\}$

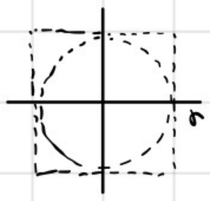
Dolžine stranic $[a,b]$: $b_j - a_j$

Volumen kvadra: $V_n([a,b]) = \prod_{i=1}^{n} (b_i - a_i)$

$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ je kvadrat $K_\infty(0,r)$



Ali sta topologiji iz d_2 in d_∞ ekvivalentni?



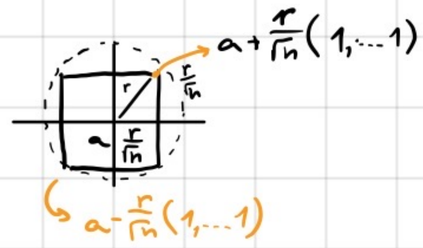
$K_\infty(0,r) \supseteq K_2(0,r)$

$\|x\|_2 < r$

$\sqrt[n]{x_1^2 + \ldots + x_n^2} \Rightarrow x_j < r \;\; \forall j = 1, \ldots, n \Rightarrow \|x\|_\infty < r$.

Oz.: $\|x\|_\infty \le \|x\|_2$

Obratno:



$\boxed{\text{Dobimo: } \|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty}$

Opomba: Metrična prostora $(\mathbb{R}^n, d_2)$ in $(\mathbb{R}^n, d_\infty)$ imata isto topologijo

Zaporedja v $\mathbb{R}^n$:

$a: \mathbb{N} \to \mathbb{R}^n$

$a(m) = a_m = (a_1^m, a_2^m, \dots, a_n^m)$

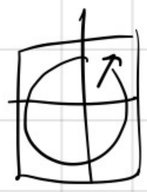
↳ zaporedje v $\mathbb{R}^n$ — n zaporedji v $\mathbb{R}$

Trditev: Naj bo (a_m) zaporedje v $\mathbb{R}^n$. $a_m = (a_1^m, \dots, a_n^m)$. Zaporedje (a_m) konvergira $\Leftrightarrow$ ko konvergirajo zaporedja $(a_1^m)_m, \dots, (a_n^m)_m$. V primeru konvergence velja:

$$\lim_{m\to\infty} a_m = \left(\lim_{m\to\infty} a_1^m, \dots, \lim_{m\to\infty} a_n^m\right)$$

Dokaz: ($\Rightarrow$): Naj (a_m) konvergira. Naj velja $\lim_{m\to\infty} a_m = a = (a_1, \dots, a_n)$.
Naj bo $\varepsilon > 0$. Potem $\exists m_0$ da za $m \geq m_0$: $d_2(a_m, a) = \sqrt{(a_1^m - a_1)^2 + \dots + (a_n^m - a_n)^2} < \varepsilon$.
$\Rightarrow \forall j = 1, \dots, n \quad \forall m \geq m_0$ velja: $|a_j^m - a| < \varepsilon$. $\Rightarrow \lim_{m\to\infty} a_j^m = a_j$.

($\Leftarrow$): Vemo: $\lim_{m\to\infty} a_j^m = a_j \quad \forall j$ in to obstaja. Naj bo $\varepsilon > 0$. Za $j = 1, \dots, n$ obstaja m_j, da za $m \geq m_j$ velja $|a_j^m - a_j| < \frac{\varepsilon}{\sqrt{n}}$.
Če je $m_0 = \max\{m_1, \dots, m_n\}$ in $m \geq m_0$, potem je $d(a_m, a) = \sqrt{(a_1^m - a_1)^2 + \dots + (a_n^m - a_n)^2} \leq \sqrt{n \cdot \left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \varepsilon$.



Zveznost preslikav iz $\mathbb{R}^n$ v $\mathbb{R}^m$

Opomba: če je $m = 1$, običajno govorimo o funkcijah

Definicija: Naj bo $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ preslikava. Naj bo $a \in D$. Preslikava f je zvezna v a, če $\forall \varepsilon > 0$ obstaja tak $\delta > 0$, da iz $\underset{x \in D}{d_2(x,a) < \delta} \Rightarrow d_2(f(x), f(a)) < \varepsilon$.

Definicija: f je <u>zvezna na D</u>, če je zvezna v vsaki točki $a \in D$.

Definicija: f je <u>enakomerno zvezna na D</u>, če $\forall \varepsilon > 0$ obstaja $\delta > 0$ da za $x, x' \in D$ velja: $d_2(x, x') < \delta \Rightarrow d_2(f(x), f(x')) < \varepsilon$.

Poskusi sam?

Opomba: Velja karakterizacija zveznosti v točki z zaporedji.

Naj bo $K \subseteq \mathbb{R}^n$ kompaktna in $f: K \to \mathbb{R}$ zvezna funkcija. Zavzame minimum, omejena maksimum, je enakomerno zv.

Trditev: Naj bosta $f, g: D \to \mathbb{R}$ funkciji. Naj bosta zvezni v $a \in D$. Tedaj so tudi $(f+g)$, $f-g$, λf, $f \cdot g$ tudi zvezne.
Če $g(x) \neq 0 \ \forall x \in D$, je tudi $\frac{f}{g}$ zvezna v a.

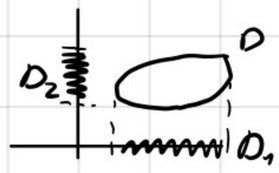
Dokaz: Kot pri analizi 1 oz. lahko z zaporedji.

- $\pi_j(x_1, \dots, x_n) = x_j \qquad |x_j - a_j| \leq d(x_j, a)$
  $(x_1, \dots, x_n) \mapsto x_j$
  } Zvezna na $\mathbb{R}^n \ \forall j$.
- Vsi polinomi v n-spremenljivkah so zvezne funkcije na $\mathbb{R}^n$.
- Vse racionalne funkcije so zvezne povsod, razen tam, kjer je imenovalec 0.

$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ je funkcija n-spremenljivk. Za $x = (x_1, \dots, x_n)$ je $f(x)$ vrednost f v x.
Velja: $f((x_1, \dots, x_n)) = f(x_1, \dots, x_n)$

Naj bo f zvezna funkcija.
$f|_N$ je potem tudi zvezna funkcija. Zožitve so vedno zvezne.

Naj bo $D \subseteq \mathbb{R}^n$. $D_j := \pi_j(D)$



Trditev: Naj bo $a \in D \subseteq \mathbb{R}^n$. Naj bo $f: D \to \mathbb{R}$ zvezna v $a = (a_1, \dots, a_n)$. Tedaj je za vsak $j = 1, \dots, n$ $\varphi_j : D_j \to \mathbb{R}$

$$\varphi_j(t) = f(a_1, \dots a_{j-1}, t, a_{j+1}, \dots, a_n) \text{ zvezna v } a_j$$

Dokaz: Naj bo $\varepsilon > 0$. Ker je f zv. v a, obstaja tak $\delta > 0$, da iz $\|x-a\| < \delta$ in $x \in D \Rightarrow |f(x) - f(a)| < \varepsilon$.
Naj bo zdaj $t \in D_j$, da je $|t - a_j| < \delta \Rightarrow \|(a_1, \dots, a_{j-1}, t, a_{j+1}, \dots, a_n) - (a_1, \dots, a_n)\| < \delta$.
Ker je f zvezna, je torej $|f(a_1, \dots, a_{j-1}, t, a_{j+1}, \dots a_n) - f(a_1, \dots, a_n)| < \varepsilon$

$$|\varphi_j(t) - \varphi_j(a_j)| < \varepsilon \Rightarrow \varphi_j \text{ je zvezna v } a. \quad \blacksquare$$

Če je funkcija več spremenljivk zvezna v nekem $a \in \mathbb{R}^n$, je v tej točki zvezna tudi kot funkcija posameznih spremenljivk.

Zožitve so zvezne

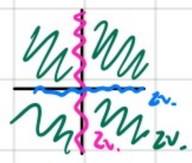
Kaj pravi trditev: Če je funkcija zvezna, so funkcije ene od komponent tudi zvezne. Spodnji primeri nam pokažejo, da obrat ne velja: funkcije so po komponentah zvezne, v celoti pa niso.

Zgled: (obrat v splošnem ne velja)

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & ; x,y \neq 0 \\ 0 & ; (x,y) = (0,0) \end{cases}$$

f je sigurno zvezna na $\mathbb{R}^2 \setminus \{(0,0)\}$. Za vse točke $(a,b) \neq (0,0)$ je po trditvi zvezna kot funkcija vsake spremenljivke posebej: $x \mapsto f(x,b)$, $y \mapsto f(a,y)$

Kaj pa v $(0,0)$? $\left.\begin{array}{l} x \mapsto f(x,0) = 0 \\ y \mapsto f(0,y) = 0 \end{array}\right\}$ zvezna



$$f(x,x) = \begin{cases} \frac{2x^2}{2x^2} = 1 & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Ta funkcija ni zvezna v $(0,0)$, torej f ni zvezna kot funkcija dveh spremenljivk.

Če $x = r\cos\varphi$, $y = r\sin\varphi$ $\quad x^2 + y^2 = r^2$, $2xy = 2r^2 \sin\varphi\cos\varphi$ $\Rightarrow f(x,y) = \begin{cases} \sin 2\varphi & ; r \neq 0 \\ 0 & ; r = 0 \end{cases}$

Zgled: $f(x,y) = \begin{cases} \frac{2x^2 y}{x^4 + y^2} & ; (x,y) \neq (0,0). \\ 0 & \text{sicer} \end{cases}$ f je zvezna na $\mathbb{R}^2 \setminus \{(0,0)\}$.

i) $x = 0$: $f(0,y) = 0$ $\forall y$, je zvezna.

ii) $y = kx$; $k \in \mathbb{R}$. $f(x,kx) = \begin{cases} \frac{2x^2 \cdot kx}{x^4 + k^2x^2} & ; x \neq 0 \\ 0 & x = 0 \end{cases} = \begin{cases} \frac{2kx}{x^2 + k^2} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ zvezna na $\mathbb{R}$

Ali je zvezna kot f. dveh spremenljivk?

$f(x,x^2) = \begin{cases} \frac{2x^4}{2x^4} = 1 & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \Rightarrow$ ni zvezna. $\Rightarrow \lim_{x,y \to 0} f(x,y) = 0$ čeprav $f(0,0) = 0$. $\Rightarrow f$ ni zvezna v $(0,0)$.

Preslikave iz $\mathbb{R}^n \to \mathbb{R}^m$

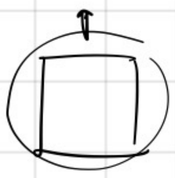
$D \subseteq \mathbb{R}^n$ in $F: D \to \mathbb{R}^m$

$$x \in D \mapsto F(x) \in \mathbb{R}^m = (y_1, \dots, y_m)$$

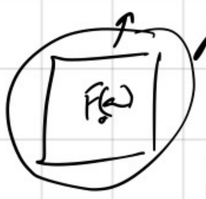
$F(x) = (f_1(x), \dots, f_m(x))$. F določa m funkcij n-spremenljivk.

Trditev: Naj bo $a \in D \subseteq \mathbb{R}^n$, $F := (f_1, \dots, f_m): D \to \mathbb{R}^m$.
F je zvezna v $a \Leftrightarrow f_1, \dots, f_m$ so zvezne v a.

Dokaz: Naj bo F zvezna v a: naj bo $\varepsilon > 0$, obstaja $\delta > 0$, da iz $x \in D$, $|x - a| < \delta \Rightarrow |F(x) - F(a)| < \varepsilon$.



Torej: $\sqrt{(f_1(x) - f_1(a))^2 + \dots + (f_m(x) - f_m(a))^2} < \varepsilon \Rightarrow \forall j$ velja: $|f_j(x) - f_j(a)| < \varepsilon$.
$\Rightarrow f_j$ je zvezna v a $\forall j = 1, \dots, m$.



Denimo, da so $f_1, \dots, f_m$ zvezne v a. Naj bo $\varepsilon > 0$. Za $\forall j = 1, \dots, m$ obstaja $\delta_j > 0$, da velja:

$$\left.\begin{array}{l} x \in D \\ |x-a| < \delta_j \end{array}\right\} \Rightarrow |f_j(x) - f_j(a)| < \frac{\varepsilon}{\sqrt{m}}$$

Naj bo $\delta := \min\{\delta_1, \dots, \delta_m\} > 0$. Za $x \in D$ in $|x-a| < \delta \Rightarrow |F(a) - F(x)| = \sqrt{(f_1(x)-f_1(a))^2 + \dots + (f_m(x)-f_m(a))^2} = \sqrt{m \cdot \left(\frac{\varepsilon}{\sqrt{m}}\right)^2} = \varepsilon$. ■

Linearne preslikave so omejene:

$A: \mathbb{R}^n \to \mathbb{R}^m$ je lin. preslikava, potem obstaja $M \geq 0$, da $\forall x \neq 0$ velja: $\frac{|Ax|}{|x|} \leq M < \infty$. (različni normi)

$\sup\limits_{x \neq 0} \frac{|Ax|}{|x|} = \sup\limits_{|x|=1} |Ax| = \|A\|$ Norma na matrikah.

Vemo, da je linearna preslikava zvezna v točki 0.

$\forall \varepsilon \ (\varepsilon = 1) \ \exists \delta > 0$, da iz $|x| < \delta \Rightarrow |Ax| < 1 \quad (|x-0| < \delta \Rightarrow |Ax - A \cdot 0| < \varepsilon)$

$x \neq 0$, $v = \frac{\delta}{2} \cdot \frac{x}{\|x\|}$, $|v| = \frac{\delta}{2}$, zato $\Rightarrow |Av| = |A \frac{\delta}{2} \frac{x}{\|x\|}| < 1 \Rightarrow \frac{|Ax|}{|x|} < \frac{2}{\delta} = M$.

Še obratno: Če je A omejena, je zvezna v 0.

$$\left.\begin{array}{l} \frac{|Ax|}{|x|} \leq M \ ; x \neq 0 \\ |Ax| \leq |x| \cdot M \quad \forall x \end{array}\right\} \Rightarrow \lim_{x \to 0} |Ax| = 0 \Rightarrow A \text{ je zvezna v } 0, \text{ zato je povsod}$$

$|Ax - Ay| = |A(x-y)|$ $\quad \lim\limits_{k \to \infty} x_k = x$. Potem je $Ax_k - Ax = A(x_k - x)$ (proti 0 ker zvezna v 0.)

$\lim\limits_{k \to 0} |Ax_k - Ax| = \lim\limits_{k \to 0} A(x_k - x) = 0$.

Naj bo $A: \mathbb{R}^n \to \mathbb{R}^m$ linearna. Tedaj je A zvezna na $\mathbb{R}^n$. Zveznost linearnih preslikav je ekvivalentna zveznosti v točki 0, kar pa je ekvivalentna omejenosti linearne preslikave. Ampak $\exists M \geq 0$, da je $\frac{|Ax|}{|x|} \leq M \Leftrightarrow$ obstaja $\sup \frac{|Ax|}{|x|} = \sup\limits_{|x|=1} |Ax| = \|A\|$. (kompakt)

(Zato so vse lin. preslikave zvezne.)

$x \mapsto Ax + b$ je afina preslikava.

$A: \mathbb{R}^n \to \mathbb{R}^m$, linearna

$b \in \mathbb{R}^m$

Parcialni odvodi in diferenciabilnost (limita, dobra afina preslikava)

Parcialni odvodi

$$\left.\begin{array}{l} f(x,y) = 2x^2 y \\ f_x(x,y) = 4xy \qquad f_y(x,y) = 2x^2 \end{array}\right\} \text{primer}$$

$a = (a_1, \dots, a_n)$

$(x_1, a_2, \dots, a_n) \in D$; $a_1 - r < x_1 < a_1 + r$

$(a_1, \dots, a_{n-1}, x_n) \in D$; $a_n - r < x_n < a_n + r$

Naj bo $a \in D$ notranja točka, torej obstaja $r > 0$, da je $K(a,r) \subseteq D$.

Naj bo $f: D \to \mathbb{R}$ funkcija.

Definicija: Funkcija f je parcialno odvedljiva po spremenljivki x_j v točki $a = (a_1, \dots, a_n) \in D$, če obstaja

$$\lim_{h \to 0} \frac{f(a_1, \dots, a_j + h, a_{j+1}, \dots a_n) - f(a_1, \dots, a_j, \dots, a_n)}{h}$$

oz. če je funkcija $x_j \mapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ odvedljiva v točki a_j.

Če ta limita obstaja, je to parcialni odvod funkcije f po spremenljivki x_j v točki a.

Oznake: $\frac{\partial f}{\partial x_j}(a)$, $f_{x_j}(a)$, $(D_{x_j} f)(a)$

Opomba: Vse elementarne funkcije so parcialno odvedljive po vseh spremenljivkah, kjer so definirane.

Zgled: $f(x,y,z) = e^{x+2y} + \cos(xz^2)$

$f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) = e^{x+2y} - \sin(xz^2) \cdot z^2$

$f_y(x,y,z) = \frac{\partial f}{\partial y}(x,y,z) = 2e^{x+2y} + 0$

$$f_z(x,y,z) = \frac{\partial f}{\partial z}(x,y,z) = 0 - \sin(xz^2)\cdot 2xz$$

Diferenciabilnost

$f: D \to \mathbb{R}$, $a \in D \subseteq \mathbb{R}^n$ naj bo notranja točka

Definicija: Funkcija f je <u>diferenciabilna</u> v točki a (notranja), če obstaja tak linearen funkcional $L: \mathbb{R}^n \to \mathbb{R}$, da velja:

$$f(a+h) = f(a) + L(h) + \sigma(h)$$

Kjer je: $\lim\limits_{h\to 0} \frac{|\sigma(h)|}{h} = 0$ / $\forall \varepsilon > 0\ \exists \delta > 0.\ 0 < |h| < \delta \Rightarrow \frac{|\sigma(h)|}{|h|} < \varepsilon.$

Opomba: Če tak L obstaja, je enolično določen.

Denimo, da za linearna funkcionala L_1 in L_2 velja $f(a+h) = f(a) + L_1(h) + \sigma_1(h)$ in $f(a+h) = f(a) + L_2 h + \sigma_2(h)$, kjer velja $\lim\limits_{h\to 0} \frac{|\sigma_1(h)|}{|h|} = 0$ in $\lim\limits_{h\to 0} \frac{|\sigma_2(h)|}{|h|} = 0$.

$\Rightarrow L_1 h + \sigma_1(h) = L_2 h + \sigma_2(h)$

$\Rightarrow \overbrace{(L_1 - L_2)}^{L} h = \sigma_2(h) - \sigma_1(h) = \sigma(h)$, velja: $\lim\limits_{h\to 0} \frac{|\sigma(h)|}{|h|} = 0$. (*)

$\Rightarrow Lh = \sigma(h) \overset{?}{\Rightarrow} L = 0$

Naj bo $v \in \mathbb{R}^n$, $v \neq 0$. $v_0 = \frac{v}{|v|}$, $|v_0| = 1$, $|v| = t \Rightarrow v = t\cdot v_0$

$L(tv_0) = \sigma(tv_0) \Rightarrow L(v_0) = \frac{\sigma(tv_0)}{t}$

$\lim\limits_{t\to 0} \frac{|\sigma(tv_0)|}{|t|} = \lim\limits_{t\to 0} \frac{|\sigma(tv_0)|}{|tv_0|} \overset{*}{=} 0 \Rightarrow Lv_0 = 0 \Rightarrow Lv = 0 \Rightarrow L \equiv 0$ oz. $L_1 = L_2$

Če je f diferenciabilna v a, je L natanko določen in ga imenujemo diferencial/odvod funkcije f v točki a. $L = df_a$ (torej odvod f v točki a - $(Df)(a)$)

$$f(a+h) = f(a) + (df_a)(h) + \sigma(h)$$

$$h \mapsto f(a) + (df_a)(h) + \sigma(h)$$

Ta afina funkcija je najboljša afina aproksimacija funkcije $h \mapsto f(a+h)$.

za:

$$f(x) = f(a) + (df_a)(x-a) + \sigma(x-a), \quad \lim_{x\to a} \frac{|\sigma(x-a)|}{|x-a|} = 0$$

Trditev: Naj bo $f: D \to \mathbb{R}$ diferenciabilna v notranji točki $a \in D$. Tedaj je f v a parcialno odvedljiva po vseh spremenljivkah. Poleg tega je zvezna v a. Pri tem velja, da je

$$(df_a)(h) = \frac{\partial f}{\partial x_1}(a)h_1 + \dots + \frac{\partial f}{\partial x_n}(a)h_n = f_{x_1}(a)h_1 + \dots + f_{x_n}(a)h_n \ ; \ h = (h_1, \dots, h_n)$$

Opomba: i) $L: \mathbb{R}^n \to \mathbb{R}$

$$Lx = l_1x_1 + \dots + l_nx_n = [l_1, \dots, l_n]\cdot\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$(df_a)(h) = \left[\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right]\cdot\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} =$$

pretvorimo vektor

$$\underbrace{\left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right)}_{\text{gradient funkcije } f \text{ v točki } a.} \cdot (h_1, \dots, h_n)$$

Oznaka: gradient f v točki a: $(\operatorname{grad} f)(a)$ ali $(\vec{\nabla} f)(a)$

$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}a, \dots, \frac{\partial}{\partial x_n}a\right)$ - operator nabla

Dokaz: f je dif. v a, torej obstaja $L: \mathbb{R}^n \to \mathbb{R}$, da je $f(a+h) = f(a) + Lh + \sigma(h)$

$\Rightarrow \lim\limits_{h\to 0} f(a+h) = \lim\limits_{h\to 0} f(a) + L(h) + \sigma(h) = f(a) + \lim\limits_{h\to 0} Lh + \lim\limits_{h\to 0} \sigma(h) = f(a)$ in f je zvezna v a.

$L = [l_1, \dots, l_n]$

$h = (h_1, 0, \dots, 0)$. $\quad h_1 \neq 0$

Velja: $f(a+h) = f(a) + Lh + \sigma(h)$

$= f(a) = l_1 h_1 + \sigma((h_1, 0, \dots, 0)) \quad / : h_1$

$$\frac{f(a+h) - f(a)}{h_1} = l_1 + \frac{\sigma(h_1(1,0,0\dots 0))}{h_1}$$

$$\lim_{h_1 \to 0} \frac{|\sigma(h_1(1,0,\dots 0))|}{|h_1(1,0,\dots 0)|} = 0 \Rightarrow \lim_{h_1 \to 0} \frac{f(a+h) - f(a)}{h_1} = l_1 \quad \frac{\partial f}{\partial x_1}(a)$$

Podobno: $l_j = \frac{\partial f}{\partial x_j}(a)$ ∎

Kaj pa obrat?

Zgled: i) $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{sicer} \end{cases}$

$x \mapsto f(x,0)$ je odvedljiva
$y \mapsto f(0,y)$ je odvedljiva
$\Big\}$ $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$ obstaja

Ampak: f ni zvezna v $(0,0)$ $\Rightarrow$ ni diferencirljiva

ii) $f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{sicer} \end{cases}$

f je gotovo zvezna na $\mathbb{R}^2 \setminus \{(0,0)\}$.

Zveznost v $(0,0)$: $x = r\cos\varphi$, $y = r\sin\varphi$ $\Rightarrow f(r\cos\varphi, r\sin\varphi) = \begin{cases} 2r\cos^2\varphi\sin\varphi & ; r > 0 \\ 0 & , r = 0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y) = (0,0) = f(0,0) \Rightarrow f \in C(\mathbb{R}^2)$
↳ je zvezna.

$x \mapsto f(x,0) \equiv 0$
$y \mapsto f(0,y) \equiv 0$
$\Big\}$ f je parcialno odvedljiva po obeh spremenljivkah v vsaki točki.

Če bi bila diferencirljiva v $(0,0)$, bi bil $df_{(0,0)} = [\underset{\frac{\partial f}{\partial x}(0,0)}{0}, \underset{\frac{\partial f}{\partial y}(0,0)}{0}]$, torej bi moralo veljati

$f(h,k) = \underset{0}{f(0,0)} + \underset{0}{df_{(0,0)}(h,k)} + \sigma(h,k)$

$\Rightarrow f(h,k) = \sigma(h,k)$

Moralo bi veljati: $\lim_{h,k\to 0} \frac{|f(h,k)|}{\sqrt{k^2+h^2}} = 0$

$\lim_{r\downarrow 0} \frac{|f(r\cos\varphi, r\sin\varphi)|}{r} = \lim_{r\downarrow 0} |2\cos^2\varphi\sin\varphi|$ limita ne obstaja, niti ni 0.

$\Rightarrow$ ni diferencirljiva v $(0,0)$

Izrek: Naj bo $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ funkcija in naj bo $a \in D$ notranja točka. Denimo, da je f parcialno odvedljiva po vseh spremenljivkah v okolici točke a in so zvezne v a.
Tedaj je f diferencirljiva v a, diferencial je sestavljen iz parcialnih odvodov.

Dokaz: $n = 2$: premik: (h,k) h,k sta dovolj majhna, da je f definirana v $(a+h, b+k)$
točka: (a,b)
koordinate: (x,y)

$\lim_{(h,k)\to 0} \frac{|\sigma(h,k)|}{\sqrt{h^2+k^2}}$

$$f(a+h, b+k) - f(a,b) \overset{?}{=} f_x(a,b)h + f_y(a,b)\cdot k + \sigma(h,k)$$

$= f(a+h, b+k) - f(a,b) = \underbrace{f(a+h, b+k) - f(a, b+k)} + \underbrace{f(a, b+k) - f(a,b)}$

2x Lagrange $\Rightarrow f_x(a^*, b+k)h + f_y(a, b^*)k$; a^* med a in $a+h$, b^* med b in $b+k$.

Ker sta f_x in f_y zvezna v (a,b): $f_x(a^*, b+k) = f_x(a,b) + \eta_1(h,k)$
$f_y(a, b^*) = f_y(a,b) + \eta_2(h,k)$

Velja: $\lim_{h,k\to 0} \eta_j(h,k) = 0$ (zveznost)

$f(a+h, b+k) - f(a,b) = f_x(a,b)h + \eta_1(h,k)h + f_y(a,b)k + \eta_2(h,k)k$ Ali je $\eta_1 + \eta_2 = \sigma$??

$$\lim_{h,k\to 0,0} \frac{|\eta_1(h,k)h + \eta_2(h,k)k|}{\sqrt{h^2+k^2}} \leq \lim_{h,k\to 0} |\eta_1(h,k)| \underset{\leq 1}{\frac{|h|}{\sqrt{h^2+k^2}}} + \lim_{h,k\to 0} |\eta_2(h,k)| \underset{\leq 1}{\frac{|k|}{\sqrt{h^2+k^2}}} = 0$$ ∎

Višji parcialni odvodi

Naj bo $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ funkcija. Naj bo D odprta podmnožica. Denimo, da je f parcialno odvedljiva po vseh spremenljivkah na D. $f_{x_1}, \dots, f_{x_n}$. Te so tudi funkcije n spremenljivk in morda so tudi te parcialno odvedljive po vseh oz. nekaterih spremenljivkah. S tem dobimo višje odvode.

Zgled: $f(x,y) = xy^2 + e^{2x+3y}$

$f_x(x,y) = y^2 + 2e^{2x+3y}$ $\quad (f_x)_x(x,y) = 4e^{2x+3y}$ $\quad (f_y)_x(x,y) = 2y + 6e^{2x+3y}$

$f_y(x,y) = 2xy + 3e^{2x+3y}$ $\quad (f_x)_y(x,y) = 2y + 6e^{2x+3y}$ $\quad (f_y)_y(x,y) = 2x + 9e^{2x+3y}$

$\Rightarrow$ Mešani odvodi so enaki

$$\left.\begin{aligned} ((f_x)_y)_y(x,y) &= 2 + 18e^{2x+3y} \\ &\parallel \\ ((f_y)_y)_x(x,y) &= 2 + 18e^{2x+3y} \end{aligned}\right]$$

($f(x,y)$ je diferencirabilna v (a,b), če obstaja $L: \mathbb{R}^2 \to \mathbb{R}$ lin. da je $f(a+h, b+k) = f(a,b) + L\left(\begin{bmatrix} h \\ k \end{bmatrix}\right) + \sigma\left(\begin{bmatrix} h \\ k \end{bmatrix}\right)$; $\lim_{k,h \to 0} \frac{|\sigma([\begin{smallmatrix} h \\ k \end{smallmatrix}])|}{|[\begin{smallmatrix} h \\ k \end{smallmatrix}]|} = 0$.

$L = df_{(a,b)}$, $L = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}](a,b) = (\text{grad } f)(a,b)$

Če je diferencirabilna, je tudi parcialno odvedljiva, obrat ne velja.

Če je zvezna kot celota, je zvezna v vsaki spremenljivki, obrat ne velja.

Če je parc. od. v točki zvezni, je dif v tej okolici.)

$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in D$. $\quad$ a je notranja!

Obstajajo $\frac{\partial f}{\partial x_i}$ $\forall i$. Ali obstajajo drugi, višji možni odvodi: $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right)$

Trditev: Naj bo f definirana v okolici $a \in \mathbb{R}^n$. Naj bosta $i,j \in \{1, \dots, n\}$ in denimo, da na tej okolici obstajata $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}$, ter obstajata tudi druga odvoda $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ in $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$.
Če sta $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ in $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$ zvezna v točki a, potem sta enaka v <u>točki a</u>.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(a).$$

Opomba: Naj bo $D \subseteq \mathbb{R}^n$ odprta. Če za drugo odvoda $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ in $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ velja, da sta zvezna na D, sta na D enaka.

- $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial i}$ (zapis za drugo odvode)

- $D_j = \frac{\partial f}{\partial x_j}$ (operator parcialnega odvajanja, po x_j). Tedaj je $D_i D_j = D_j D_i$ (linearni)
  Ti operatorji komutirajo

- Kerda obstajajo odvodi tretjega/višjih redov. Kdaj velja, da je: $i,j,k \in \{1, \dots, n\}$.
  $D_j D_i D_k = D_i D_k D_j = D_k D_i D_j = D_k D_j D_i \dots$
  To bo veljalo, ko so vsi tretji parcialni odvodi zvezni v a, so potem enaki v a.
  Podobno za še višje odvode.

- Naj bo $D \subseteq \mathbb{R}^n$ odprta. Naj bo $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ funkcija. Če je f parc. odvedljiva na D in so ti parc. odvodi zvezni na D, potem je f v vsaki $a \in D$ diferencirabilna.

- Če je f na D dvakrat parc. odvedljiva po vseh spremenljivkah in so drugi parcialni odvodi zvezni na D, potem ti komutirajo. $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ itd.

- $C^{(0)}(D)$ je algebra zveznih funkcij na D.

- $C^1(D)$ (obstajajo vsi parc. odvodi in so vsi zvezni) 1x parci. zvezno odvedljive funkcije.

$\frac{\partial f}{\partial x_i} \in C^1(D) \quad \forall i$

- $C^k(D)$ je prostor vseh funkcij, ki so po vseh spremenljivkah k-krat parcialno odvedljive. Ti odvodi so potem zvezne funkcije na D.

- $C(D) \supseteq C^1(D) \supseteq C^2(D) \ldots$
  $C^k(D)$ je vek. prostor, tudi algebra nad $\mathbb{R}$.

- $\bigcap_1^\infty C^i(D) = C^\infty(D)$ so gladke funkcije

„dovolj: n=2, mi zapišemo

Dokaz: $J = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

i) $\varphi(x) = f(x, b+k) - f(x, b)$

$J = \varphi(a+h) - \varphi(a)$ $\quad$ f je parc. odv. po obeh $\Rightarrow$ φ je odvedljiv, zato, na φ uporabimo Lagrangeev izrek.

Uporabimo Lagrangeev izrek na φ.

$J = \frac{d\varphi}{\partial x}(a^*)\, h$; a^* med a in $a+h$.

$= \left(\frac{\partial f}{\partial x}(a^*, b+k) - \frac{\partial f}{\partial x}(a^*, b)\right) h$

$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(a^*, b^*)\right) h \cdot k$; b^* med b in $b+k$

ii) $\psi(y) = f(a+h, y) - f(a, y)$

$J = \psi(b+k) - \psi(b) = f(a+h, b+k) - f(a, b+k) - f(a+h, b) - f(a,b)$

Drugič Lagrangeev izrek:

$J = \frac{d\psi}{dy}(b^{**})\, k$; b^{**} med b in $b+k$.

$= \left(\frac{\partial f}{\partial y}(a+h, b^{**}) - \frac{\partial f}{\partial y}(a, b^{**})\right) k$

$\overset{L.I.}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(a^{**}, b^{**})\right) h \cdot k$; a^{**} leži med a in $a+h$.

$J = J = \ldots \quad \forall k, h \neq 0.$

$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(a^*, b^*)\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(a^{**}, b^{**})\right)$

$h, k \to 0 \Rightarrow a^* \to a \quad b^* \to b$

$a^{**} \to a \quad b^{**} \to b$

Ker sta oba odvoda zvezna v (a,b), dobimo enakost. ■

Diferenciabilnost preslikav:

$F: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, a v notranjosti D.

F je diferenciabilna v a, če obstaja taka linearna preslikava $L: \mathbb{R}^n \to \mathbb{R}^m$, da velja: $F(a+h) = F(a) + L(h) + o(h)$, kjer je $\lim_{h \to 0} \frac{|o(h)|_m}{|h|_n} = 0$.

Opomba: kot pri funkcijah, če L obstaja, je enolično določen.

L je diferencial F v točki a oz. dF_a, je tudi odvod F v a: $(DF)(a)$.

Zgled: $A: \mathbb{R}^n \to \mathbb{R}^m$ linearna.

$F(x) = Ax$

$a \in \mathbb{R}^n$, $F(a+h) = A(a+h) = Aa + Ah = F(a) + Ah + 0$

$dF_a = A \quad \forall a.$

Zgled: $F: \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$

$X \mapsto X^2$, $F(X) = X^2$

$A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times n}$. $F(A+H) = (A+H)^2 = A^2 + HA + AH + H^2 = F(A) + (AH+HA) - H^2$

Če diferencial obstaja, kaj bi bil kandidat?

$(DF)(A)H = AH + HA$ je lin. preslikava $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ v H.

$\sigma(H) = H^2$??

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $\quad \|A\| = \sqrt{a_{11}^2 + \dots + a_{nn}^2}$ (vsi členi)

Velja: $|AB| \le |A||B|$. Če to velja, potem je $0 \le \lim_{H\to 0} \frac{|H^2|}{|H|} \le \lim_{H\to 0} \frac{|H|^2}{|H|} = 0$

$$\underset{A}{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}} \underset{B}{\begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}} \Rightarrow |(A\cdot B)_{ij}|^2 = |\vec{a_i}\cdot\vec{b_j}|^2 \le |\vec{a_i}|^2 |\vec{b_j}|^2$$

$$|AB|^2 = \sum_{i,j} (AB)_{ij}^2 \le \sum_i \sum_j |a_i|^2 |b_j| = \sum_i |\vec{a_i}|^2 \cdot |B|^2 = |A|^2 \cdot |B|^2$$

Zgled: $F(x,y,z) = (x^2 + 2y + e^z, xy + z^2)$

$F: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad (1,0,1)$??

Izrek: Naj bo $a \in D$ notranja točka. Naj bo $F = (f_1, \dots, f_m): D \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Preslikava F je diferenciabilna v $a \in D$ $\Leftrightarrow$ $f_1, \dots, f_m$ so diferenciabilne v a.

Tedaj je diferencial F v a $(DF)(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \dots, & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}, & \dots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(a)$ $\quad m\times n$ matrika, $\mathbb{R}^n \to \mathbb{R}^m$, v vrsticah so gradienti.

$$(DF)(x,y,z) = \begin{bmatrix} 2x & 2 & e^z \\ y & x & 2z \end{bmatrix}$$

$$(DF)(1,0,1) = \begin{bmatrix} 2 & 2 & e \\ 0 & 1 & 2 \end{bmatrix}$$

<u>Posledica</u>: Če so vse funkcije $f_1, \dots, f_m$ v okolici a parc. odvedljive po vseh spremenljivkah in so ti odvodi zvezni v a, potem je F diferenciabilna v a.

Dokaz izreka: ($\Rightarrow$): Naj bo F diferenciabilna v a. $\exists\, dF_a = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$.

$F(a+h) = F(a) + (dF_a)(h) + \sigma(h)$

Zapis po komponentah.

$f_1(a+h) = f_1(a) + \underbrace{A_{11}h_1 + \dots + A_{1n}h_n}_{\text{linearno v } h} + \sigma_1(h)$ $\quad$ ↳ to mora biti majhno.

Ali velja: $\lim_{h\to 0} \frac{|\sigma_1(h)|}{|h|} = 0$?

Vemo: $\lim_{h\to 0} \frac{|\sigma(h)|_m}{|h|_n} = 0 = \lim_{h\to 0} \frac{\sqrt{\sigma_1(h)^2 + \dots + \sigma_m(h)^2}}{|h|_n} = 0$

$$\Downarrow$$

$$\lim_{h\to 0} \frac{|\sigma_1(h)|_m}{|h|_n} = 0 \quad \checkmark$$

f_1 je diferenciabilna, diferencial pa je $[A_{11}, \dots, A_{1n}]$, $A_{ij} = \frac{\partial f_i}{\partial x_j}(a)$

($\Leftarrow$): Naj bodo $f_1, \dots, f_m$ diferenciabilne v a. $\quad A_{ji} = \frac{\partial f_j}{\partial x_i}$

Torej: $f_j(a+h) = f_j(a) + A_{j1}h_1 + \dots + A_{jn}h_n + \sigma_j(h)$ $\quad j = 1, \dots, m$

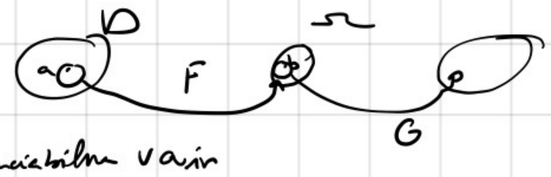
$F(a+h) = (f_1(a+h), \dots, f_m(a+h))$

$= F(a) + \underset{\text{kin.}}{Ah} + \sigma(h)$; $\sigma(h) = (\sigma_1(h), \dots, \sigma_m(h))$

$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$

Ali spet velja: $\lim_{h\to 0} \frac{|\sigma(h)|_m}{|h|_n} = 0 = \lim_{h\to 0} \frac{|\sigma(h)|_m}{|h|_n} = \lim_{h\to 0} \frac{\sqrt{\sigma_1^2(h) + \dots + \sigma_m^2(h)}}{|h|_n} = \lim_{h\to 0} \sqrt{\left(\frac{|\sigma_1(h)|}{|h|}\right)^2 + \dots + \frac{|\sigma_m(h)|^2}{|h|^2}} = 0$

Opomba: F je razreda C^k, če so $f_1, \dots, f_m \in C^k$.



Verižno pravilo:

Izrek: Naj bo $a \in D \subseteq \mathbb{R}^n$ notranji. Naj bo $b \in \Omega \subseteq \mathbb{R}^m$ notranji. Naj bo $F: D \to \Omega$ diferenciabilna v a in velja $F(a) = b$. Naj bo $G: \Omega \to \mathbb{R}^k$ diferenciabilna v b. Tedaj je $G \circ F$ diferenciabilna v a in velja:

$$\underset{k\times n}{D(G\circ F)(a)} = \underset{k\times m}{(DG)(b)} \cdot \underset{m\times n}{(DF)(a)} = (DG)(F(a)) \cdot (DF)(a).$$

$$F(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$
$$G(y_1,\dots,y_m) = (g_1(y_1,\dots,y_m),\dots,g_k(y_1,\dots,y_m))$$

$$D(G\circ F)(a) = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial g_k}{\partial y_1} & \cdots & \frac{\partial g_k}{\partial y_m} \end{bmatrix}(b) \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(a)$$

Posledica: $k=1$, $G=g$ - funkcija:

$$\Phi(x_1,\dots,x_n) = g(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$
$$\frac{\partial \Phi}{\partial x_j}(a) = \frac{\partial g}{\partial y_1}(b)\cdot\frac{\partial f_1}{\partial x_j}(a) + \frac{\partial g}{\partial y_2}(b)\frac{\partial f_2}{\partial x_j}(a) + \dots + \frac{\partial g}{\partial y_m}(b)\frac{\partial f_m}{\partial x_j}(a)$$

Zgled: $F(x,y) = (\overset{u}{x^2+y}, \overset{v}{xy})$

$g(u,v) = uv + v^2$

$\Phi := g\circ F$

$\Phi(x,y) = g(F(x,y)) = g(x^2+y, xy) = (x^2+y)(xy) + x^2y^2 = x^3y + xy^2 + x^2y^2$

$\frac{\partial \Phi}{\partial x}(x,y) = 3x^2y + y^2 + 2xy^2$

$\frac{\partial \Phi}{\partial y}(x,y) = x^3 + 2xy + 2x^2y$ $\qquad \frac{\partial g}{\partial u} = v \quad \frac{\partial g}{\partial v} = u+2v$

$$\frac{\partial \Phi}{\partial x}(x,y) = \frac{\partial}{\partial x}(g\circ F)(x,y) = \frac{\partial g}{\partial u}(F(x,y))\cdot\frac{\partial f_1}{\partial x}(x,y) + \frac{\partial g}{\partial v}F(x,y)\cdot\frac{\partial f_2}{\partial x}(x,y)$$
$$= (xy)\cdot 2x + (x^2+y)\cdot 2xy\cdot y$$
$$= 2x^2y + x^2y + y^2 + 2xy^2$$
$$= 3x^2y + y^2 + 2xy^2$$

$$\frac{\partial \Phi}{\partial y}(x,y) = \frac{\partial g}{\partial u}F(x,y)\,\frac{\partial f_1}{\partial y}(x,y) + \frac{\partial g}{\partial v}(F(x,y))\,\frac{\partial f_2}{\partial y}(x,y)$$
$$= (xy)\cdot 1 + (x^2+y+2xy)(x)$$
$$= x^3 + 2xy + 2x^2y$$

Dokaz (izreka): F je dif. v a. $F(a+h) = F(a) + \overbrace{(DF)(a)h + \sigma_F(h)}^{b}$

G je dif. v b. $G(b+k) = G(b) + (DG)(b)k + \sigma_G(k)$

$$(G\circ F)(a+h) = G(F(a+h)) = G(F(a)) + DG(F(a))\big((DF)(a)h + \sigma_F(h)\big) + \sigma_G\big((DF)(a)h + \sigma_F(h)\big)$$
$$= (G\circ F)(a) + (DG)(b)DF(a)h + \sigma(h)$$

$$\sigma(h) = (DG)(b)\sigma_F(h) + \sigma_G\big((DF)(a)h + \sigma_F(h)\big)$$

Ali $\lim_{h\to 0}\frac{|\sigma(h)|}{|h|} = 0$?

Vemo, da sta $(DG)(b)$ in $(DF)(a)$ omejeni, ker sta zvezni.

Obstaja $M_{1,2} > 0$. $\forall v \in \mathbb{R}^n$ in $\forall w \in \mathbb{R}^m$: $|DF(a)v|_m \le M_1|v|_n$ in $|(DG)(b)w|_k \le M_2|w|_m$

$$0 \le \lim_{h\to 0}\frac{|(DG)(b)\sigma_F(h)|}{|h|} \le \lim_{h\to 0} M\frac{|\sigma_F(h)|}{|h|} = 0$$

Ostane še:

$$\lim_{h\to 0} \frac{|\sigma_G((DF)(a)h+\sigma_F(h))|}{|h|} = 0.$$

Vemo: $\lim_{k\to 0} \frac{|\sigma_G(k)|}{|k|} = 0$

Naj bo $\varepsilon>0$. Potem obstaja $\delta>0, 0<|k|<\delta \Rightarrow \frac{|\sigma_G(k)|}{|k|}<\varepsilon \Leftrightarrow \forall k, |k|<\delta: |\sigma_G(k)| \le \varepsilon\cdot|k|$.

Vemo tudi: $\lim_{h\to 0}[(DF)(a)h+\sigma_F(h)]=0$

Zato obstaja δ_1: $\forall h: |h|<\delta_1 \Rightarrow [(DF)(a)h+\sigma_F(h)] \le \delta |h|$

$\Rightarrow$ Velja $|\sigma_G((DF)(a)h+\sigma_F(h))| \le \varepsilon\, |(DF)(a)h+\sigma_F(h)| \le \varepsilon\cdot(M\cdot|h|+|\sigma_F(h)|)$

$$\frac{|\sigma_G((DF)(a)h+\sigma_F(h))|}{|h|} \le \varepsilon\Big(\overbrace{M_1 \frac{|\sigma_F(h)|}{|h|}}^{\text{omejeno}}\Big)$$

$$\Rightarrow \lim_{h\to 0} \frac{|\sigma_G((DF)(a)h+\sigma_F(h))|}{|h|} = 0$$

$\exists M_1$ končno, da za $|h|<\delta_1$: $\left(M_1 \frac{\sigma_F(h)}{|h|}\right) < M_1$ ■

Izrek o implicitni funkciji: $Ax=b$; A je linearna.

$A: \mathbb{R}^n \to \mathbb{R}^m$.

i) $m \le n$

ii) rang $A = m$

$2x+3y=1$

$y=\frac{1}{3}(1-2x)=\varphi(x)$

$ax+by=c \overset{b\neq 0}{\Rightarrow} y=\frac{1}{b}(c-ax)=\varphi(x)$

Kaj bo: $y=\varphi(x)$ pri $x^2+y^2=a$?

okoli $(0,1)$: $y=\sqrt{1-x^2}$ okoli $(1,0)$? To ni graf nad x, je pa nad y.

okoli $(0,-1)$: $y=-\sqrt{1-x^2}$ okoli $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$? To je graf nad obema osema.

Če $a<0$ ni rešitev v $\mathbb{R}$.

$f(a,b)=0$

Radi bi radostno pogledali na $f(x,y)$, da bo enačba $f(x,y)=0$ lokalno v okolici (a,b) predstavljala graf funkcije $y=\varphi(x)$.

$\alpha x+\beta y=\gamma$ bo graf nad x-osjo $\Leftrightarrow \beta\neq 0$.

Želimo si: $f_y(a,b)\neq 0$.

$f(a+x,b+y)=f(a,b)+f_x(a,b)x+f_y(a,b)y+o(x,y)$

Izrek (osnovna verzija izreka o implicitni funkciji):

Naj bo D odprta $\subseteq \mathbb{R}^2$. Naj bo $(a,b)\in D$. Naj bo $f\in C^1(D)$ in naj velja:

i) $f(a,b)=0$

ii) $f_y(a,b)\neq 0$

Potem obstajata $\delta>0$ in $\varepsilon>0$, da velja:

$I\times J \subseteq D$; $I=(a-\delta,a+\delta)$, $J=(b-\varepsilon,b+\varepsilon)$ (pravokotnik okoli (a,b))

in enolično določena C^1 funkcija $\varphi: I\to J$, za katero velja:

i) $\varphi(a)=b$

ii) $\forall (x,y)\in I\times J$ je $f(x,y)=0 \Leftrightarrow y=\varphi(x)$. rešitve enačbe zvestovaj pravokotniku so natanko graf φ.

$f(x,\varphi(x))=0 \quad \forall x$

$f_x\cdot 1+f_y\cdot\varphi'=0 \Rightarrow \varphi'(x)=-\frac{f_x(x,\varphi(x))}{f_y(x,\varphi(x))} \quad \forall x\in I.$

Zgled: $f(x,y)=x^2+y^2-1$

$V\ (\overset{a}{0},\overset{b}{1})$: $\quad f_y(x,y) = 2y \qquad f_y(0,1) = 2 \neq 0$

Zgled: $\quad x^2+y^2+xy^3+x^2y^4+x^5y^5=1.$

$f(x,y) = x^2+y^2+xy^3+x^2y^4+x^5y^5-1$

$f(0,1)=0$

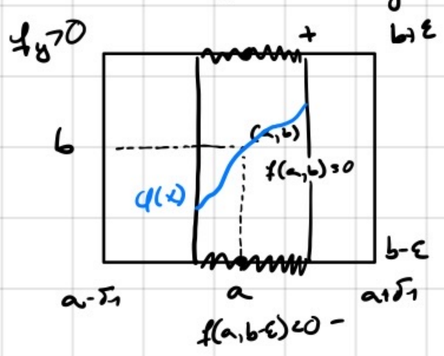
$f_y(0,1)=2\neq 0$

Dobim $\varphi(x)$, $\varphi(0)=1$, vsi enačbo lokalno

Dokaz izreka: $\quad f_y(a,b)\neq 0$

BŠS je $f_y(a,b)>0$

Trditev: Obstajata $\delta_1,\varepsilon>0$, da je $f_y>0$ na $[a-\delta_1,a+\delta_1]\times[b-\varepsilon,b+\varepsilon]\subseteq D$.



$\forall x\in[a-\delta_1,a+\delta_1]$ je $y\mapsto f(x,y)$ strogo naraščajoča

$\Rightarrow f(a,b-\varepsilon)<0$ in $f(a,b+\varepsilon)>0$

$\exists\, 0<\delta\leq\delta_1$, da je $\forall x\in[a-\delta,a+\delta]$ $f(x,b-\varepsilon)<0$ in $f(x,b+\varepsilon)<0$.

Ker je f zvzna $\forall x\in[a-\delta,a+\delta]$ obstaja natanko ena točka $\varphi(x)=y\in(b-\varepsilon,b+\varepsilon)$

da velja $f(x,\varphi(x))=0$

Trdimo, da je $\varphi\in C^1((a-\delta,a+\delta))$

$f_y>0$ na $[a-\delta,a+\delta]\times[b-\varepsilon,b+\varepsilon]$, je kompakt.

$\exists\, M>0$, da je $f_y(x,y)>M>0$ na tem pravokotniku

$x\in(a-\delta,a+\delta)$

Δx ; $x+\Delta x\in(a-\delta,a+\delta)$

$y=\varphi(x)$

$y+\Delta y=\varphi(x+\Delta x)$

$\left(\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\overset{?}{=}\varphi'(x)\right)$

Velja: $\quad f(x+\Delta x,y+\Delta y)-\overset{0}{f(x,y)}=0=\overbrace{f(x+\Delta x,y+\Delta y)-f(x+\Delta x,y)}^{2x\ \text{Lagrange}}+\overbrace{f(x+\Delta x,y)-f(x,y)}^{}=$

$\Rightarrow f_y(x+\Delta x,y^*)\Delta y+f_x(x^*,y)\Delta x=0$

y^* med y in $y+\Delta y$, x^* med x in $x+\Delta x$

$$\Rightarrow \Delta y=\frac{-f_x(x^*,y)}{f_y(x+\Delta x,y^*)}\Delta x$$

omejeno na kompaktu $I\times J$, ker je $f\in C^1(D)$ in so odvodi zvezni, omejeni

Ko $\Delta x\to 0$, tudi $\Delta y\to 0$ $\Rightarrow$ zvezna

$\hookrightarrow \lim\limits_{\Delta x\to 0}\Delta y=\lim\limits_{\Delta x\to 0}-\frac{f_x(x^*,y)}{f_y(x+\Delta x,y)}\Delta x=0 \Rightarrow \varphi$ je zvezna v x, torej povsod na $(a-\delta,a+\delta)$.

Odvedljivost: $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}-\frac{f_x(x^*,y)}{f_y(x+\Delta x,y^*)}=\frac{-f_x(x,y)}{f_y(x,y)}$

Zaključi, je φ' zv.: $\varphi'(x)=\frac{-f_x(x,\varphi(x))}{f_y(x,\varphi(x))}$ $\to$ kompozitum zveznih preslikav.

$\Rightarrow \varphi'$ je zv. oz. $\varphi\in C^1(I)$ $\blacksquare$

Rešitev $f(x,\varphi(x))$ je graf $\varphi(x)$

Opomba: Če je $f\in C^k(D)\Rightarrow\varphi\in C^k(I)$.

$k=2$: $\varphi'(x)=\frac{-f_x(x,\varphi(x))}{f_y(x,\varphi(x))}$

$$\left.\begin{array}{l}\varphi\in C^1(I)\\ f\in C^2(D),\ f_x,f_y\in C^1(D)\end{array}\right\}\Rightarrow\varphi'\in C^1\Leftrightarrow\frac{-f_x(x,\varphi(x))}{f_y(x,\varphi(x))}\in C^1\Rightarrow\varphi\in C^2(I)$$

Za vsak k dobimo z indukcijo.

(Opomba: $y'=y$ - rešitev mora biti C^∞)

Zgled: i) $f(x,y) = (x-y)^2$

$f(x,y) = 0 \iff x = y$

$(a,b) = (0,0) \quad f(0,0) = 0$

$f_y(x,y) = (-2)(x-y) \Rightarrow f_y(0,0) = 0$ (pogoji niso izpolnjeni in imamo lepo rešitev)

ii) $f(x,y) = y^3 - x$ v $(0,0) = (a,b)$.

$f(0,0) = 0$

$f_y(0,0) = 3y^2$ v $(0,0) = 0$

$f(x,y) = 0 \iff y = \sqrt[3]{x} \rightarrow$ to je graf neke funkcije, a ni odvedljiva v $x = 0$.

iii) $f(x,y) = y^2 - x^2 - x^4$

$f(0,0) = 0$

$f_y(x,y) = 2y \quad f_y(0,0) = 0$ (Niso izpolnjeni zadostni pogoji)

$f(x,y) = 0 \Rightarrow y^2 = x^2 + x^4 \quad \left.\begin{array}{l} y_1 = x\sqrt{1+x^2} \\ y_2 = -x\sqrt{1+x^2} \end{array}\right\}$ imamo več rešitev - ni enoličnosti

iv) $f(x,y) = y^2 + x^2 + x^4$

$f(0,0) = 0$

$f_y(0,0) = 0$

Edina rešitev je $(0,0)$, nimamo grafa

Splošni izrek o implicitni funkciji:

$n+m$ spremenljivk (x,y); $x = (x_1, \dots, x_n)$

m enačb: $y = (y_1, \dots, y_m)$

$\hookrightarrow f_1(x,y) = 0$

$\vdots$ Radi bi $y = \Phi(x)$

$f_m(x,y) = 0$

Linearni primer: $Ax + By = b$

$A: \mathbb{R}^n \to \mathbb{R}^m$

$B: \mathbb{R}^m \to \mathbb{R}^m$

$b \in \mathbb{R}^m$.

Kdaj (glede na A, B) lahko za vsak b iz te enačbe dobimo y kot funkcijo x?

$\iff \det B \neq 0. \iff$ obstaja B^{-1}

Ker: $y = B^{-1}(b - Ax)$

Poseben primer: $n = 0$: $By = b$ lahko rešimo enolično $\forall b \in \mathbb{R}^m \iff \det B \neq 0$

$m \in \mathbb{N}$, $\mathbb{R}^m$. $D^{odp} \subseteq \mathbb{R}^m$, $\Phi: D \to \mathbb{R}^m$, $\Phi \in C^1(D) \rightarrow$ komponente so iz C^1.

$a \in D$, $b = \Phi(a)$.

Kakšni so zadostni pogoji za lokalno obrnljivost Φ?

Opomba: $B: \mathbb{R}^m \to \mathbb{R}^m$ lin. bijekcija, tudi B^{-1} obstaja in je linearen.

Definicija: Naj bo $\Phi: D^{odp} \subseteq \mathbb{R}^m \to \Omega^{odp} = \Phi(D) \subseteq \mathbb{R}^m$ C^1 in bijekcija. Preslikava Φ je difeomorfizem, če je tudi $\Phi^{-1} \in C^1(\Omega)$

$\Phi: D \to \Omega$ je C^1 difeomorfizem, če: i) Φ je bijekcija

ii) Φ je $C^1(D)$

iii) Φ^{-1} je $C^1(\Omega)$

Podobno definiramo C^k difeomorfizem, $k \in \mathbb{N} \cup \{\infty\}$.

Zgled: Ali iz i, ii $\Rightarrow$ (iii)

$f(x) = x^3$, bijekcija $\quad f'(0) = 0 \rightarrow$ problem!

$f: \mathbb{R} \to \mathbb{R}$, $f \in C^\infty(\mathbb{R})$

$f^{-1}(x) = \sqrt[3]{x}$, ni odvedljiva v $x = 0 \Rightarrow$ ni difeomorfizem.

Trditev: Naj bo $\Phi: D^{odp} \to \Omega^{odp}$ C^1 difeomorfizem. Tedaj je $\det(D\Phi) \neq 0$ na D.

$\det(D\Phi) = \det(d\Phi)$

2. $D\Phi$ je v vsaki točki D obrnljiva matrika.

Dokaz: $\exists \Phi^{-1}: \Omega \to D$, $\Phi^{-1} \in C^1(\Omega)$.

$\Phi^{-1} \circ \Phi = Id_D$

Po verižnem pravilu: $(D\Phi^{-1})(\Phi(x)) \cdot (D\Phi)(x) = I_{n\times n} \Rightarrow \det \neq 0$.

$\Rightarrow \det(D\Phi) \neq 0$ na D. ■

Dodatek: $(D\Phi^{-1})(y) = (D\Phi)^{-1}(x)$, $y = \Phi(x)$.

Zgled: $\Phi(x,y): \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(x,y) = (e^x \cos y, e^x \sin y)$

- $\Phi(x, y+2k\pi) = \Phi(x,y)$ $\forall k \in \mathbb{Z}$. Ni injektivna, niti sur.
- $(D\Phi)(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$

$\det(D\Phi)(x,y) = e^{2x} \neq 0$

Imamo izpolnjen potreben pogoj za difeomorfizem, ki pa očitno ni zadosten.

Φ ni difeo. ker ni niti bijekcija

Če bi iskali inverz: $e^x \sin y = v$

$e^x \cos y = u$

$e^{2x} = u^2 + v^2$

$e^x = \sqrt{u^2+v^2}$

$x = \ln(\sqrt{u^2+v^2})$

Če si omejimo za y, bi lahko tudi y dobili - lokalno bi lahko uspeli.

Izrek (izrek o inverzni preslikavi):

Naj bo $D \overset{odp}{\subseteq} \mathbb{R}^n$ in $\Phi: D \to \mathbb{R}^n$ C^1 preslikava. Naj bo $a \in D$ in $b = \Phi(a)$.

Če je $\det(D\Phi)(a) \neq 0 \Leftrightarrow$ je odvod obrnljiv, obstajata okolici $a \in U \subseteq \mathbb{R}^n$, $b \in V \subseteq \mathbb{R}^n$, da je $\Phi: U \to V$ C^1 difeomorfizem.

Opomba: Φ je <u>lokalni</u> difeomorfizem

Dodatek: Če je $\Phi \in C^k(D)$, $k \in \mathbb{N} \cup \{\infty\}$, je Φ lokalni C^k difeomorfizem.

Opomba: $m=1$: $f: I \subseteq \mathbb{R} \to \mathbb{R}$, $a \in I$, C^1

$f'(a) \neq 0$

V okolici je strogo naraščajoča/padajoča - ima lokalni C^1 inverz

Opomba: Dodatek sledi iz izreka (C^1 verzija) - podobno sklep kot prej:

$(D\Phi^{-1})(y) = (D\Phi)^{-1}(x)$; $\Phi(x) = y$.

$(D\Phi^{-1})(y) = (D\Phi)^{-1}(\Phi^{-1}(y))$

↳ racionalna funkcija v elementih matrike

Naj bo $k=2$: $\Phi \in C^2 \Rightarrow$ tudi C^1. Po izreku je Φ lokalni C^1 difeomorfizem.

$(D\Phi^{-1})(y) = \underbrace{(D\Phi)^{-1}}_{C^1,\ \det(D\Phi)\neq 0 \text{ v okolici } a}(\overset{\to C^1}{\Phi^{-1}(y)}) \Rightarrow$ kompozitum C^1 preslikav $\Rightarrow (D\Phi^{-1})$ je $C^1 \Rightarrow \Phi^{-1} \in C^2$.

Za ostale k dobimo z indukcijo.

Poenostavitev: $\Phi: D \to \mathbb{R}^n$ $\quad x \mapsto \Phi(x+a) - \Phi(a)$

$a \mapsto b$

Lahko predpostavimo $a = b = 0$.

$x \mapsto \Phi(x)$, $0 \mapsto 0$.

$x \mapsto (D\Phi)^{-1}(0) \circ \Phi(x) = \Psi(x)$

$\Psi(0) = 0$, $(D\Psi)(0) = (D\Phi)^{-1}(0)\,(D\Phi)(0) = I$.

Izrek je dovolj dokazati za primer $a = b = 0$ in $(D\Phi)(0) = I$.

Izrek o implicitni preslikavi:

(Dokazali ga bomo s pomočjo izreka o inverzni preslikavi)

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$.

$(a,b) \in D^{odp} \subseteq \underset{x}{\mathbb{R}^n} \times \underset{y}{\mathbb{R}^m}$

$F: D \to \mathbb{R}^m$ C^1

1.) $F(a,b) = 0$ 2.)

$F(x,y) = 0 \iff y = \varphi(x)$, $x \approx a$, $y \approx b$.

$\varphi: U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, C^1.

$F(x,y)$, $\frac{\partial F}{\partial x}$

Za vsak y naj bo $\frac{\partial F}{\partial x}$ diferencial preslikave $x \mapsto F(x,y)$

Za vsak x naj bo $\frac{\partial F}{\partial y}$ diferencial preslikave $y \mapsto F(x,y)$

Če je F C^1 preslikava, je $\frac{\partial F}{\partial x}(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$, $m \times n$ dimenzij

$$\frac{\partial F}{\partial y}(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{bmatrix}$$

Kako izgleda popolni odvod? $(DF)(x,y) = \left[\underset{n}{\frac{\partial F}{\partial x}(x,y)}, \underset{m}{\frac{\partial F}{\partial y}(x,y)} \right]_m^{(n+m)\times m}$

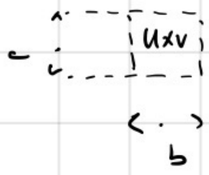
$$(DF)(x,y)\begin{bmatrix} h \\ k \end{bmatrix} = \frac{\partial F}{\partial x}(x,y)h + \frac{\partial F}{\partial y}(x,y)k \in \mathbb{R}^m.$$

$$\frac{\partial F}{\partial x}(a,b)x + \frac{\partial F}{\partial y}(a,b)y = c \approx Ax + By = c$$

$\hookrightarrow \det \neq 0$

Izrek o implicitni preslikavi:

Naj bo $D^{odp} \subseteq \mathbb{R}^n_x \times \mathbb{R}^n_y$. Naj bo $(a,b) \in D$ $(a \in \mathbb{R}^n, b \in \mathbb{R}^m)$. Naj bo $F: D \to \mathbb{R}^m$ C^1 preslikava in naj velja:



i) $F(a,b) = 0$

ii) $\det\left(\frac{\partial F}{\partial y}(a,b)\right) \neq 0$.

Tedaj obstaja okolica U točke a in V točke b, $a \in U \subseteq D$, $b \in V \subseteq \mathbb{R}^m$ in taka enolično določena C^1 preslikava $\varphi: U \to V$, da velja:

k, odvedljivost se podeduje

i) $\varphi(a) = b$

ii) $\forall (x,y) \in U \times V$ je $F(x,y) = 0 \iff y = \varphi(x)$.

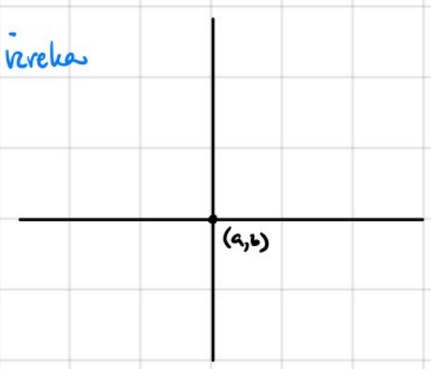
Rešitve te enačbe = graf $\varphi(x)$ znotraj $U \times V$.

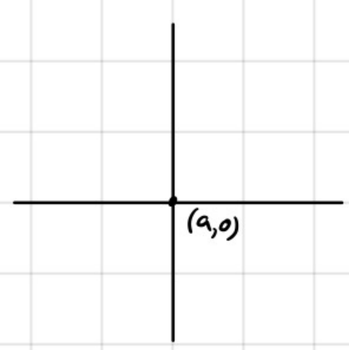
iii) $(D\varphi)(x) = -\left(\frac{\partial F}{\partial y}(x,y)\right)^{-1}\left(\frac{\partial F}{\partial x}(x,y)\right)$; $y = \varphi(x)$

<u>Dodatek</u>: če je F C^k je tudi φ C^k.

<u>Opomba</u>: $F(x, \varphi(x)) = 0$

Odvajamo po x: $\frac{\partial F}{\partial x}(x,\varphi(x)) + \frac{\partial F}{\partial y}(x,\varphi(x)) \cdot (D\varphi)(x)$ } dobimo (iii) iz izreka





Dokaz (s pomočjo I. o inverzni preslikavi):

Definiramo preslikavo $\Phi: (x,y) \in D \subseteq \mathbb{R}^n \times \mathbb{R}^m \mapsto (x, F(x,y)) \subseteq \mathbb{R}^n \times \mathbb{R}^m$.

$(a,b) \mapsto (a,0)$

Odvod:

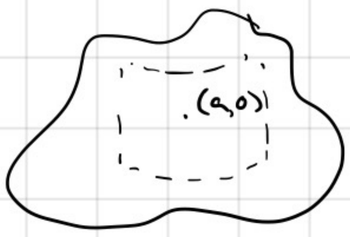
$$\begin{bmatrix} I_{n\times n} & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}(a,b) \Rightarrow \det \begin{bmatrix} I_n & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}(a,b) = \det\left(\frac{\partial F}{\partial y}(a,b)\right) \neq 0 \text{ po predpostavki.}$$

$\Rightarrow$ Preslikava $\Phi: (x,y) \mapsto (x, F(x,y))$ je iz okolice (a,b) v okolico $(a,0)$ lokalni difeomorfizem in ima C^1 inverz.

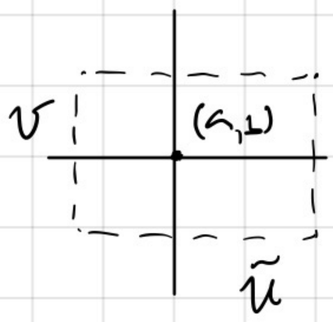
(Obstaja $\mathcal{U} \ni (a,b)$, $\mathcal{V} \ni (a,0)$, da je $\Phi: \mathcal{U} \to \mathcal{V}$ C^1 difeomorfizem - bijekcija in ima C^1 inverz)

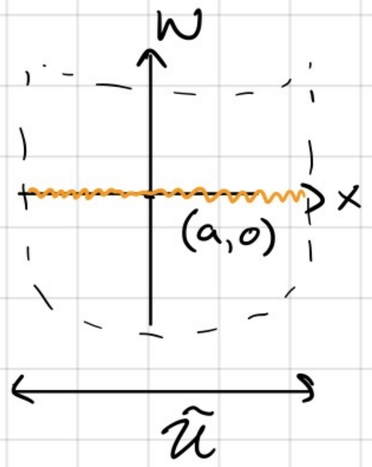
$\hookrightarrow$ dvog kot v traditi

BŠS: $\mathcal{U} = \tilde{\mathcal{U}} \times V$



$\Phi: \tilde{\mathcal{U}} \times V \to \mathcal{V}$ je C^1 dif.





$\Phi^{-1}(x,w) = (x, G(x,w))$ nehaj pač

Kandidat za $\varphi(x) = G(x,0)$

$a \in U \subseteq \mathbb{R}^n$, $U = \{x \in \mathbb{R}^n \mid (x,0) \in V\}$, $\varphi(x) = G(x,0) : U \to V$.

i) $\underline{\varphi(a) = b}$

$\varphi(a) = G(a,0)$

$F(a,b) = 0$ po predpostavki

$\Phi(a,b) = (a,0) \Rightarrow \Phi^{-1}(a,0) = (a,b)$

$\underset{\varphi(a)}{\underset{\shortparallel}{(a, G(a,0))}} = \underset{b}{\underset{\shortparallel}{(a,b)}} \Rightarrow \underline{\varphi(a) = b}.$

ii) 1) Naj bo $(x,y) \in U \times V$, $F(x,y) = 0$

Naj velja $\Phi(x,y) = (x,0)$

Zato $\underset{(x, G(x,0))}{\underset{\shortparallel}{\Phi^{-1}(x,0)}} = (x,y)$

$\Rightarrow \varphi(x) = G(x,0) = y$

2) Naj bo $y = \varphi(x) \overset{?}{\Rightarrow} F(x,y) = 0$?

$y = G(x,0)$

$\left.\begin{array}{l} \Phi(x,y) = (x, F(x,y)) \\ \Phi^{-1}(x,0) = (x, G(x,0)) = (x,y) \end{array}\right\} \Phi(x,y) = (x,0) = (x, F(x,y)) \Rightarrow \underline{F(x,y) = 0}$

iii) Sledi iz (ii), le odvajamo kompozitum. ■

Zgled: $F(x,y) = x^2 + y^2 - 1$

$(a,b) = (0,1)$. $F(0,1) = 0$ in $\frac{\partial F}{\partial y}(0,1) = 2 \neq 0$ $\frac{\partial F}{\partial y} = 2y$, velko por.

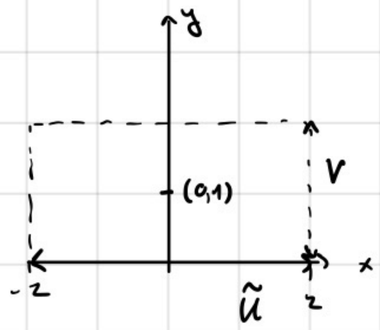
$\Phi(x,y) = (x, x^2+y^2-1) = (u,w)$

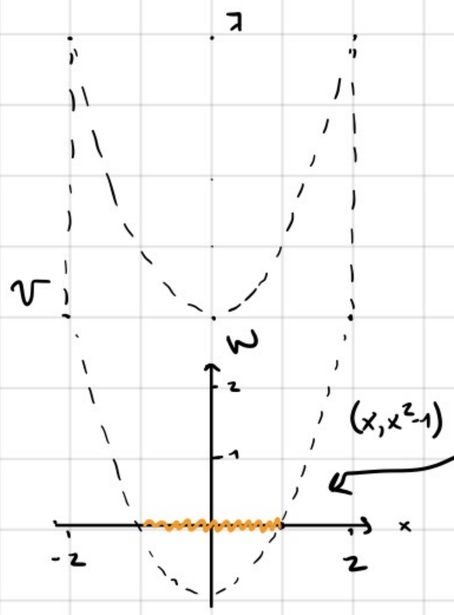
Inverz: $x = u$

$y = \sqrt{w+1-x^2}$ $\Phi^{-1}(x,w) = (x, \sqrt{w+1-x^2})$

$w = 0$

$\varphi(x) = \sqrt{1-x^2}$, definirano $|x| < 1$, to je U.





Zgled: $\left.\begin{array}{l} f(x,y,z) = y + xy + xz^2 = 0 \\ g(x,y,z) = z + zy + x^2 = 0 \end{array}\right\}$

$(0,0,0)$ je rešitev

$$\begin{bmatrix} \frac{\partial f}{\partial y} = 1+x & \frac{\partial g}{\partial y} = z \\ \frac{\partial f}{\partial z} = 2xz & \frac{\partial g}{\partial z} = 1+y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

V $(0,0,0)$ je

$(-\varepsilon,\varepsilon)$

$\Rightarrow$ Obstaja okolica $x=0$ v $\mathbb{R}$

$\left.\begin{array}{l}\varphi(x)=y(x)\\ \psi(x)=z(x)\end{array}\right\}$ sta C^∞ funkciji za kateri velja:

$y(0)=0$ (ker je $(0,0,0)$ rešitev enačbe

$z(0)=0$

$\forall x\in(-\varepsilon,\varepsilon)$ je $f(x,y(x),z(x))=0$

$g(x,y(x),z(x))=0$

y,z sta funkciji x

$y+xy+xz^2$ odvajamo po x: $y'+y+xy'+z^2+2xzz'=0$

$z+zy+x^2$ odvajamo po x: $z'+z'y+zy'+2x=0$

V $x=0$ $(y=0, z=0)$: $y'(0)=0$

$z'(0)=0$

Odvajamo še enkrat: $y''+y'+xy''+y'+2zz'+2zz'+2xz'^2+2xzz''=0$

$x=0 \Rightarrow y,z,y',z'=0 \Rightarrow \underline{\underline{y''(0)=0}}$

$z''+z''y+z'y'+z'y'+zy''+2=0$

$x=0 \Rightarrow y,z,y',z'=0 \Rightarrow \underline{\underline{z''(0)=-2}}$

Dokaz izreka o inverzni preslikavi:

$\left(F: D^{odp}\subseteq\mathbb{R}^n\to\mathbb{R}^n,\ a\in D\mapsto F(a)=b\right.$: obstaja okolica $a\in U$ in $b\in V$, da je $F: U\to V$ C^1 (ali C^k) dif.$\left.\right)$ $\quad \det((DF)(a))\neq 0$

Vemo: BSS je $a=b=0$ in $(DF)(a)=I$ $\quad x\longmapsto (DF)^{-1}(0)\,F(x)=G(x)$

Zato je $F(x)=x+H(x)$.

$(DG)(0)=(DF)^{-1}(0)\,(DF)(0)=I$

samo lin. transformacija, ki nam ohranja rodu

$F(0)=0 \Rightarrow H(0)=0$

$(DF)(0)=I+(DH)(0)=I \quad \Rightarrow \quad H(0)=0$, $(DH)(0)=0$

$F(x)=x+H(x)$; $H(0)=0\in\mathbb{R}^n$ in $(DH)(0)=0\in\mathbb{R}^{n\times n}$.

Pomožna trditev: $\exists r>0\ \forall x_1,x_2\in\overline{K(0,r)}$ velja: $|H(x_2)-H(x_1)|\leq\frac{1}{2}|x_1-x_2|$ (posledica Lagrangeevega izreka)

1.) F je injektivna na $\overline{K(0,r)}$

$x_1,x_2\in\overline{K(0,r)}$

$|F(x_1)-F(x_2)|=|(x_1-x_2)+(H(x_1)-H(x_2))|\geq|x_1-x_2|-|H(x_1)-H(x_2)|\overset{\text{pomožna}}{\geq}\frac{1}{2}|x_1-x_2|$.

Zato: Če $x_1\neq x_2 \Rightarrow F(x_1)\neq F(x_2) \Rightarrow$ injektivnost

2.) $F:\overline{K(0,r)}\to F(\overline{K(0,r)})$ je bijekcija

$|F(x_1)-F(x_2)|\geq\frac{1}{2}|x_1-x_2|$

$|y_1-y_2|\geq\frac{1}{2}|F^{-1}(y_1)-F^{-1}(y_2)| \Rightarrow$ Inverzna preslikava je zvezna (tudi Lip.). - Skupaj je homeomorfizem.

3.) „Velikost" slike $F(\overline{K(0,r)})$: Trdimo, da je $\overline{K(0,\frac{r}{2})}\subseteq F(\overline{K(0,r)})$

Če $y\in\overline{K(0,\frac{r}{2})}$ ali $\exists x\in\overline{K(0,r)}: F(x)=y$?

$\Leftrightarrow x+H(x)=y \Leftrightarrow x=\underbrace{-H(x)+y}_{T_y(x)}$

$x=T_y(x)$

Iščemo negibno točko T_y na $\overline{K(0,r)}$.

i) $(\overline{K(0,r)},d_2)$ je poln metrični prostor

ii) $T_y:\overline{K(0,r)}\to\overline{K(0,r)}$ (??)

$x\in\overline{K(0,r)} \Leftrightarrow |x|\leq r$

$|T_y(x)|=|-H(x)+y|\leq|H(x)|+|y|=|H(x)-H(0)|+|y|\overset{\text{pomožna}}{\leq}\frac{1}{2}|x-0|+|y|\leq\frac{r}{2}+\frac{r}{2}=\underline{\underline{r}}$

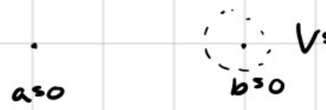
$\Rightarrow T_y:\overline{K(0,r)}\to\overline{K(0,r)}$

T_y - je skrčitev: $|T_y(x_1)-T_y(x_2)| = |H(x_1)-H(x_2)| \le \frac{1}{2}|x_1-x_2|$ $\quad (q=\frac{1}{2})$

$\Rightarrow$ T_y ima negibno točko na $\overline{K(0,r)}$, torej enačba $F(x)=y$ je rešljiva na $\overline{K(0,r)}$ $\forall y \in \overline{K(0,\frac{r}{2})}$

Opomba: Če je $|y|<\frac{r}{2}$, potem je $F(x)=y$ za neki $|x|<r$.

F je inj na $\overline{K(0,r)}$, $F(\overline{K(0,r)}) \supseteq \overline{K(0,\frac{r}{2})}$ in $F(K(0,r)) \supseteq K(0,\frac{r}{2})$, inverz od F je zvezen (tudi Lip.)



$U := K(0,r) \cap F^{-1}(V)$ - je odprta množica, hkrati pa je $F: U \to V$ bijekcija in zvezna.

BŠS je $\det(DF(x)) \neq 0$ na U $\Rightarrow$ $\forall x$ $\exists (DF)^{-1}(x)$ $\qquad (DF(0)=I,\ F\in C^1)$

Odvedljivost inverza F^{-1}: $\qquad ((DF^{-1})(y) = (DF)^{-1}(x)$, za $F(x)=y$.

$y\in V$, $k\in\mathbb{R}^n$ da je $y+k\in V$

$$\lim_{k\to 0} \frac{|F^{-1}(y+k)-F^{-1}(y)-(DF)^{-1}(x)(k)|}{|k|} \overset{?}{=} 0 \quad (\text{ali je šteje } o(k)?)$$

$$\left.\begin{array}{l} y = F(x) \Leftrightarrow x = F^{-1}(y) \\ y+k = F(x+h) \Leftrightarrow x+h = F^{-1}(y+k) \end{array}\right\} \Rightarrow \quad k = F(x+h)-F(x) = (DF)(x)(h) + o(h)$$

Vemo: $F(x+h) = F(x) + (DF)(x)(h) + o(h)$ $\quad$ (F je C^1 $\Rightarrow$ je diferenciabilna)

$$\frac{|F^{-1}(y+k)-F^{-1}(y)-(DF)^{-1}(x)k|}{|k|} = \frac{|\cancel{x}+h-\cancel{x}-(DF)^{-1}(x)\,\cancel{((DF)(x)h+o(h))}|}{|k|} = \frac{|(DF)^{-1}(x)\,o(h)|}{|k|}$$

$$\le M\frac{|o(h)|}{k} = M\underbrace{\frac{|o(h)|}{|h|}}_{\to 0}\cdot\frac{|h|}{|k|} \le 2M\frac{|o(h)|}{|h|} \xrightarrow[k\to 0]{} 0$$

Vemo, da je $|F^{-1}(\underset{y+k}{y_1})-F^{-1}(\underset{y}{y_2})| \le 2|y_1-y_2|$

$\Rightarrow |h| \le 2|k|$

$\Rightarrow \frac{|h|}{|k|} \le 2$

$\Rightarrow$ F^{-1} je diferenciabilna na V.

$F(x)=y$

$(DF^{-1})(y) = \underbrace{(DF)^{-1}}_{\text{zv.}}(\underbrace{F^{-1}(y)}_{\text{zv.}})$ $\Rightarrow$ tudi $(DF^{-1})(y)$ je zvezna $\Rightarrow$ F^{-1} je C^1. ■

Pomožna trditev: $H(0)=0$, $(DH)(0)=0$, H je C^1

$\exists r>0$, $\forall x_1,x_2 \in \overline{K(0,r)}$: $|H(x_1)-H(x_2)| \le \frac{1}{2}|x_1-x_2|$

Trditev: Naj bo $D\subseteq\mathbb{R}^n$, $a,b\in D$, takšni, $\forall t\in[0,1]$ je $ta+(1-t)b \in D$

Naj bo $f\in C^1(D)$. Tedaj obstaja točka η iz daljice med a in b, da je

$f(b)-f(a) = (Df)(\eta)(b-a)$

Dokaz: $\varphi(t) = f(ta+(1-t)b)$ $\quad$ φ je definirana in C^1 v okolici $[0,1]\subseteq\mathbb{R}$

$\varphi(1) = f(a)$

$\varphi(0) = f(b)$

Lagrange izrek za φ: $\quad \varphi(1)-\varphi(0) = \varphi'(\tau)(1-0)$; $\tau\in(0,1)$

$f(a)-f(b) = (Df)(\underbrace{\tau a+(1-\tau)b}_{\eta})(a-b)$

$\Rightarrow f(b)-f(a) = (Df)(\eta)(b-a)$

Posledica: Predpostavke kot v prejšnji trditvi. Naj $\exists M<\infty$, da $\forall j=1,\dots,n$ in $\forall x\in D$ velja: $\left|\frac{\partial f}{\partial x_j}(x)\right|<M$ (omejimo odvod)

Tedaj je $|f(b)-f(a)|\le M\cdot\sqrt{n}\,|b-a|$

Dokaz: $|f(b)-f(a)| = |(Df)(\eta)(b-a)| \overset{\text{C.S. neenakost}}{\le} |(Df)(\eta)|\cdot|b-a| = \sqrt{\underbrace{\left(\frac{\partial f}{\partial x_1}(\eta)\right)^2}_{\le M}+\dots+\underbrace{\left(\frac{\partial f}{\partial x_n}(\eta)\right)^2}_{\le M}}\,|b-a| \le M\sqrt{n}\,|b-a|$

Posledica: D, $a,b\in D$ kot prej. $F: D\subseteq\mathbb{R}^n\to\mathbb{R}^m$ razreda C^1. $F=(f_1,\dots,f_m)$

Naj obstaja $M<\infty$, da za $\forall j=1,\dots,n$ $\forall i=1,\dots,m$ in $\forall x\in D$ velja, da je $\left|\frac{\partial f_i}{\partial x_j}(x)\right|\le M$ (omejimo vse odvode)

Tedaj je

$$|F(b)-F(a)|\le M\sqrt{m\cdot n}\,|b-a|$$

Dokaz: $|F(b)-F(a)| = \sqrt{\underbrace{(f_1(b)-f_2(b))^2+\dots+(f_m(b)-f_n(a))^2}_{\le M^2\cdot n}}\,|b-a| \le M\sqrt{m\cdot n}\,|b-a|.$

Dokaz pomožne trditve:

Ker je H razreda C^1 in ker je $(DH)(0)=0^{n\times n}$ obstaja $r>0$ $\forall x\in\overline{K(0,r)}$ velja $\left|\frac{\partial H_r}{\partial x_j}\right|\le\frac{1}{2n}$

$\Rightarrow |H(x_1)-H(x_2)|\le\frac{1}{2n}\sqrt{nn}\,|x_1-x_2| = \frac{1}{2}|x_1-x_2|$

$F(x)=x+H(x)$

F injektivna: $F^{-1}: F(\overline{K(0,r)})\to\overline{K(0,r)}$ zvezna

$\overline{K(0,\frac{\varepsilon}{2})}\subseteq F(\overline{K(0,r)})$ (negibna točka) - tudi za odprte velja

$V=K(0,\frac{\varepsilon}{2})$

$U=K(0,r)\cap F^{-1}(V)$

$F: U\to V$ je bijekcija, sledi iz zvezne odvedljivosti

Zgled: $F:\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}$

$X\mapsto X^2$

$(DF)(A)H = AH+HA$

$(DF)(I_n)H = 2H$

$(DF)(I_n) = 2I_n$

$\exists U\ni I_n$ in $V\ni I_n$: $F: U\to V$ difeomorfizem, $F(I)=I$

$\forall Y\in V\ \exists!\ X\in U$, da je $F(X)=Y$ oz. $X^2=Y$

Na V lahko dobro definiramo $\sqrt[2]{Y}$. $\sqrt[2]{I}=I$

Zgled: $F(x,y,z)=0$ (rešujemo), $F:\mathbb{R}^3\to\mathbb{R}$, $F(a,b,c)=0$ (dana rešitev.)

$\frac{\partial F}{\partial z}(a,b,c)\ne 0 \Rightarrow z=f(x,y)$

$\frac{\partial F}{\partial y}(a,b,c)\ne 0 \Rightarrow y=g(x,z)$

$\frac{\partial F}{\partial x}(a,b,c)\ne 0 \Rightarrow x=h(y,z)$

Če je $(DF)(a,b,c)\ne 0$ (rang $(DF)(a,b,c)=1$) lahko vsaj eno od teh izrazimo implicitno.

Zgled: $F(x,y,z)=0$, $G(x,y,z)=0$ Vemo: $F(a,b,c)=0$ in $G(a,b,c)=0$

$$\operatorname{rang}\begin{bmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z\end{bmatrix}(a,b,c) = 2$$

2x2 poddeterminante

Obstaja par spremenljivk (x,y,z), ki jih lahko podamo kot funkcije tretje

Definicija: Naj bo $D^{odp} \subseteq \mathbb{R}^n$ in $F: \mathbb{R}^n \to \mathbb{R}^m$ C^1 preslikava. Naj bo $a \in D$.

- Rang F v točki: $\operatorname{rang}_a F := \operatorname{rang}(DF)(a)$.
- Če je $\operatorname{rang}_a F$ konstanten $\forall a \in D$, potem je F tega ranga na D: $\operatorname{rang} F = \operatorname{rang}_a F$.
- F je maksimalnega ranga v a ali na D, če je $\operatorname{rang}_a F = \min\{n,m\}$.

Opomba: ta pogoj je lokalno stabilen (na okolici velja, ne le v točki)

$F: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $\quad m \leq n.$ $\qquad m \geq n$

$(DF)(a) = \underset{n}{[\quad \square \quad]}\, m$ $\qquad (DF)(a)\ \overset{n}{[\square]}\, m$

lahko permutiramo koordinate, da dobimo obrnljive poddeterminante tukaj.

Posledica izreka o implicitni preslikavi: Naj bo $F: D^{odp} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ C^k preslikava ($k \in \mathbb{N}$) in naj velja, da je $m < n$. Naj bo $a \in D$, $F(a) = 0$ in F maksimalnega ranga v točki a. Tedaj obstajajo indeksi: $i_1 \leq \ldots \leq i_{n-m}$, $j_1 < \ldots < j_m$; $i_k \neq j_l$ $\forall k,l$ in take C^k funkcije $\varphi_1, \ldots, \varphi_m$ definirane v okolici $a_{i_1}, \ldots, a_{i_{n-m}}$, da je v neki okolici U točke a enačba $F(x) = 0$ ekvivalentna sistemu enačb:

$$x_{j_1} = \varphi_1(x_{i_1}, \ldots, x_{i_{n-m}})$$
$$\vdots$$
$$x_{j_m} = \varphi_m(x_{i_1}, \ldots, x_{i_{n-m}})$$

Ekvivalentno: obstaja permutacija $\sigma \in S_n$, da v okolici a velja: $F(x) = 0 \iff (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (\underbrace{(x_{\sigma(1)}, \ldots, x_{\sigma(n-m)})}_{x'_\sigma}, \underset{(\varphi_1, \ldots, \varphi_m)}{\varphi})$

Dokaz: $\operatorname{rang}(DF)(a) = m$

$DF(a) = \underset{n}{[\qquad]}\, m$ $\qquad$ Obstaja obrnljiva $m \times m$ podmatrika, ki je spravimo na konec s permutacijo.

Koordinate permutiramo tako, da je to zadnja $m \times m$ podmatrika. = zadnjih m stolpcev je lin. neodvisnih.

Potem uporabimo izrek o implicitni preslikavi.

$A: \mathbb{R}^n \to \mathbb{R}^m$ $\quad m \leq n$, $\operatorname{rang} A = m$.

Za A vemo, da je surjektivna, $Ax = b$ Prostor rešitev je „$(n-m)$ dimenzionalen"

Posledica: Naj bo $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ C^1 preslikava. Naj bo $m \leq n$. Naj bo $a \in D$ in naj velja:

$\operatorname{rang}_a F = m$.

Tedaj obstaja okolica $V \ni F(a) = b$ in $U \ni a$, da je $F: U \to V$ surjektivna

Dokaz: Poglejmo preslikavo $(\overset{D}{x}, \overset{\mathbb{R}^m}{y}) \overset{\Phi}{\longmapsto} F(x) - y$.

(Če je $m = n$ je F lokalni difeomorfizem in je lokalno surjektivna)

Naj bo $m < n$.

$b = F(a)$ $\quad (D\Phi)(a,b) = [(DF)(a), -I_m]$, $\Phi(a,b) = 0$

(Iz $y \approx x \overset{?}{\Rightarrow} \exists x \approx a$, da je $\Phi(x,y) = 0$)

Vemo: $\operatorname{rang}(DF)(a) = m$. Lahko permutiramo koordinate $x = (x_1, \ldots, x_n)$, da bo v matriki $DF(a)$ zadnjih m stolpcev linearno neodvisnih.

Po izreku o implicitni preslikavi lahko v okolici (a,b) neznanke $(x_{n-m+1}, \ldots, x_m)$ izrazimo kot funkcije spremenljivk $x_1, \ldots, x_{n-m}, \underbrace{y_1, \ldots, y_m}_{y}$, torej $\forall y \approx b$ $\exists x \approx a$, da je $\Phi(x,y) = 0 \iff F(x) = y$. ∎

Podmnogoterosti v $\mathbb{R}^n$:

Posplošitev pojmov krivulja in ploskev.

Definicija: Naj bo $M \neq \emptyset$. $M \subseteq \mathbb{R}^{n+m}$ je gladka (vsaj C^1) podmnogoterost dimenzije n in kodimenzije m prostora $\mathbb{R}^{n+m}$, če $\forall a \in M$ obstaja okolica $U \subseteq \mathbb{R}^{n+m}$, $a \in U$ in taka C^1 funkcija $F_1, \dots, F_m : U \to \mathbb{R}$, da velja:

i) $M \cap U = \{x \in U \mid F_1(x) = \dots = F_m(x) = 0\}$

ii) $\operatorname{rang}(F_1, \dots, F_m) = m$ na U.

Opombe: • $F_1, \dots, F_m$ se imenujejo definicijske funkcije za $M \cap U$.

• Če $m = 0$ bo M odprta množica

Zgled: $\mathbb{R}^3$: $F_1, \dots F_m$ so linearne funkcije

$F_1(x,y,z) = 0$ (ravnina, razen če je $F_1 \equiv 0$)

Kodimenzija $= 1$, dimenzija je 2.

$\boxed{\text{Če je dimenzija} = 1 \text{ govorimo o krivuljah, če je dimenzija} = 2, \text{ govorimo o ploskvah.}}$

$F_1(x,y,z) = 0$
$F_2(x,y,z) = 0$ (premica (presek ravnin), razen v izjemnih primerih - zato pogoj na rangu - F_1 in F_2 sta lin. neodvisna)

Kodimenzija $= 2$, dimenzija $= 1$

$F_1(x,y,z) = 0$
$F_2(x,y,z) = 0$ (točka, če so F_1, F_2, F_3 lin. neodvisni funkcionali, je edina rešitev $\{(0,0,0)\}$. - zato pogoj na rangu)
$F_3(x,y,z) = 0$.

Kodimenzija $= 3$, dimenzija $= 0$.

$M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \neq \emptyset$.

$F(x,y,z) = x^2 + y^2 + z^2 - 1$

$DF = (2x, 2y, 2z)$

Ali je $\operatorname{rang} F = 1$ v okolici M.

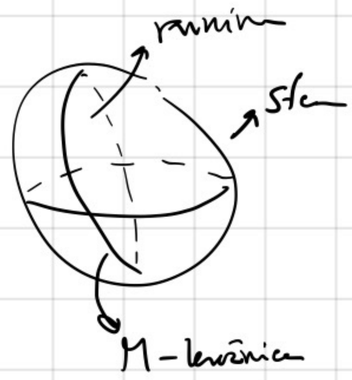
$(\operatorname{rang} F)(x,y,z) = 0 \iff x = y = z = 0$ (ni točka v M)

M je potem gladka podmnogoterost dimenzije 2 in kodimenzije 1.

$N = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x + y + z = 0\}$

$F(x,y,z) = x^2 + y^2 + z^2 - 1$

$G(x,y,z) = x + y + z$



Či je, bo to krivulja (dimenzija $= 1$)

$D(F,G) = \begin{bmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$

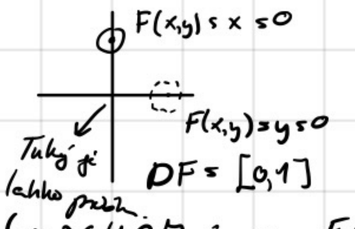
rang $D(F,G)$ ali 1 ali 2.

Ali je lahko rang na N enak 1? $\iff$ $2x = \lambda$, $2y = \lambda$, $2z = \lambda$; $\lambda \in \mathbb{R}$ $\iff$ $x = \frac{\lambda}{2} = y = z$

$x^2 + y^2 + z^2 = 1 \iff \frac{3\lambda^2}{4} = 1$
$x + y + z = 0 \iff \frac{3}{2}\lambda = 0$ } Obaja hkrati ne gre.

$\Rightarrow$ rang $D(F,G)$ na N je vedno 2. $\Rightarrow$ N je res krivulja.

$M = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$

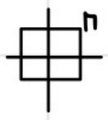


Ali obstaja okolica okoli $(0,0)$ in U.

da je $F : U \to \mathbb{R}$ in $F = 0 \iff (x,y) \in U \cap M$ in $\operatorname{rang} F(0,0) = 1$?

Če bi bil bila $F(x,0) = 0$ $x = 0 \Rightarrow \frac{\partial F}{\partial x}(0,0) = 0$
$F(0,y) = 0$ $y = 0 \Rightarrow \frac{\partial F}{\partial y}(0,0) = 0$ } $\Rightarrow$ rang ne bo 1

$\operatorname{rang} F(0,0) = 0$,

(Če bi gledali  M ni gladka mnogoterost)

Zgled: $GL_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$.

$A \mapsto \det A$ je zv.

$GL_n(\mathbb{R})$ je odprta množica v $\mathbb{R}^{n\times n}$ in je n^2-podmnogoterost.

$$SL_n(\mathbb{R}) \overset{F^{-1}(\{0\})}{=} \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$$

To je n^2-1 dimenzionalna gladka podmnogoterost v $\mathbb{R}^{n\times n}$.

$F(A) = \det A - 1$. $\quad F: \mathbb{R}^{n\times n} \to \mathbb{R}$

Ali je preslikava maksimalnega ranga?

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} A_{11} - a_{12} A_{12} + \ldots + (-1)^n a_{1n} A_{1n}$$

kofaktorji

kimaš a_{11}

$$\frac{\partial F}{\partial a_{11}} = A_{11} \qquad \frac{\partial F}{\partial a_{12}} = -A_{12} \quad \ldots \quad \frac{\partial F}{\partial a_{1n}} = (-1)^{n+1} a_{1n}$$

Če bi veljalo, da je $A_{11} = A_{12} = \ldots = A_{1n} = 0 \Rightarrow \det A = 0$ in $A \notin SL_n(\mathbb{R})$

F je zato dobra definicijska funkcija za $SL_n(\mathbb{R})$.

Opomba: $M \cap U = \{x \in U \mid F_1 = \ldots = F_m = 0\}$, $\operatorname{rang} F = m$.

Mnogoterost je podana implicitno

Lokalno je M graf

Trditev: Neprazna podmnožica $M \subseteq \mathbb{R}^{n+m}$ je podmnogoterost dimenzije n $\Leftrightarrow$ za vsako točko $a \in M$ $\exists U$ odprta okolica te točke v $\mathbb{R}^{n+m}$ in taka permutacija koordinat $\sigma: x_1, \ldots, x_{n+m} \mapsto x_{\sigma(1)} \ldots x_{\sigma(n+m)}$, da je $M \cap U$ graf neke C^1 ali C^k preslikave $\varphi: D^{odp} \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

$M \cap U = \{x \in U \mid (x_{\sigma(1)}, \ldots, x_{\sigma(n+m)}) = (x_{\sigma(1)} \ldots x_{\sigma(n)}, \varphi(x_{\sigma(1)} \ldots x_{\sigma(n)}))\}$

Opomba: Lokalno je M graf nad enim od n-dimenzionalnih koordinatnih podprostorov.

Dokaz: $\Rightarrow$: Posledica trditve iz razdelka o implicitni preslikavi.

$M \cap U$: $F_1 = 0$, $\ldots$, $F_m = 0$ $\qquad F_1, \ldots, F_m \in C^1(U)$, $\operatorname{rang}(F_1, \ldots, F_m) = m$.

$\Leftarrow$: $M \cap U = \{(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \mid (x_1, \ldots, x_n, \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n))\}$.

$F_1(x) = x_{n+1} - \varphi_1(x_1, \ldots, x_n)$

$F_2(x) = x_{n+2} - \varphi_2(x_1, \ldots, x_n)$

$\vdots$

$F_m(x) = x_{n+m} - \varphi_m(x_1, \ldots, x_n)$

$M \cap U = \{x \in U;\ F_1 = \ldots = F_m = 0\}$.

Še rang: $F = (F_1, \ldots, F_m)$

$DF = [-D\varphi, I]$, $\operatorname{rang} DF = m$ ■

Opomba: Če mnogoterost podajamo kot graf, rečemo, da je podana eksplicitno.

Zgled: $M = \{(x, x^2);\ x \in \mathbb{R}\}$

Parametrično podajanje mnogoterosti:

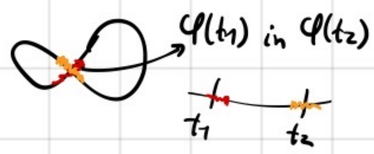
$t \mapsto (x(t), y(t)) \subseteq \mathbb{R}^2$

Zgled: $\varphi \mapsto (a\cos\varphi, a\sin\varphi)$ $\quad \varphi \in [0, 2\pi)$

Trditev: i) Naj bo $M \subseteq \mathbb{R}^{n+m}$ podmnogoterost dimenzije n. Potem za vsak $a \in M$ obstaja taka okolica $U \ni a$ v $\mathbb{R}^{n+m}$ in taka C^1 preslikava $\Phi: D^{odp} \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$ maksimalnega ranga (n), da je $\Phi(D) = M \cap U$.

ii) Naj bo $D \subseteq \mathbb{R}^n$ odprta in $\Phi: D \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$ C^1 preslikava ranga n. Potem za vsako točko $t_0 \in D$ obstaja okolica $t_0 \in V \subseteq D$, da je $\Phi(V)$ podmnogoterost dimenzije n v $\mathbb{R}^{n+m}$.

Zgled:



Dokaz: i) Lokalno je M v okolici a graf. $M \cap U = \{(x_1,\dots,x_n,x_{n+1},\dots,x_{n+m}) \in U \mid x_{n+1} = \varphi_1(x_1,\dots,x_n), \dots, x_{n+m} = \varphi_m(x_1,\dots,x_n)\}$

$\Phi(x_1,\dots,x_n) = ((x_1,\dots,x_n), \varphi_1(x_1,\dots,x_n), \dots, \varphi_m(x_1,\dots,x_n))$.

Ali je rang maksimalen? $D\Phi = \begin{bmatrix} I \\ D\varphi \end{bmatrix}_m^n$; $\varphi = (\varphi_1,\dots,\varphi_m)$ $\quad \Phi: D \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$

rang $D\Phi = n$.

ii) $\Phi: D \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$ ranga n. $t_0 \in D$.

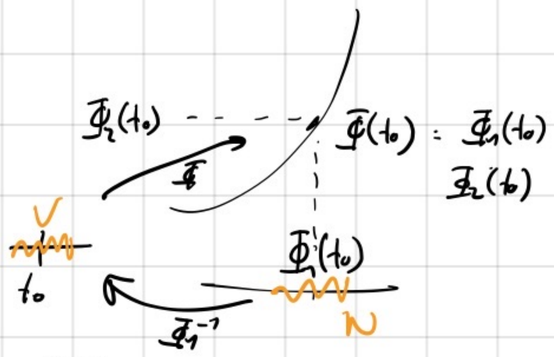
$(D\Phi)(t_0) = \begin{bmatrix} \text{obrnljiva} \\ \hline \end{bmatrix}^n_m$

S permutacijo koordinat v sliki ($\mathbb{R}^{n+m}$), da je zgornja $n \times n$ matrika obrnljiva

$\Phi = (\Phi_1, \Phi_2)$; $\Phi_1: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$

$\Phi_2: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$(D\Phi) = \begin{bmatrix} D\Phi_1 \\ D\Phi_2 \end{bmatrix}$ obrnljiva v t_0



Vemo, da je $\det(D\Phi_1)(t_0) \neq 0 \Rightarrow \Phi_1$ je lokalni difeomorfizem. $\Rightarrow \exists V$ točka t_0 in W točka $\Phi_1(t_0)$, da je $\Phi_1: V \to W$ difeomorfizem.

Obstaja inverz $\Phi_1^{-1}: W \to V$. Tvorimo $(\Phi \circ \Phi_1^{-1}) = (id, \overbrace{\Phi_2 \circ \Phi_1^{-1}}^{\varphi})$

$(\Phi \circ \Phi_1^{-1})(x_1,\dots,x_n) = (x_1,\dots,x_n, \varphi_1(x_1,\dots,x_n), \dots, \varphi_m(x_1,\dots,x_n))$

Torej M je lokalno graf nad $\mathbb{R}^n$. ■

Krivulje v $\mathbb{R}^3$

$t \xmapsto{\Phi(t)} (x(t), y(t), z(t))$

$\Phi(t) = (x(t), y(t), z(t))$

$(D\Phi)(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

Kdaj je ranga 1? $\operatorname{Rang}(D\Phi)(t) = 1 \Leftrightarrow \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \neq 0 \quad \forall t$.

Če to velja, je regularna parametrizacija.

Ploskve v $\mathbb{R}^3$: $(t,s) \xmapsto{\Phi} (x(t,s), y(t,s), z(t,s))$

$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ je lin. odvisen od $\vec{b}$. Če $\vec{a} \times \vec{b} \neq 0 \Rightarrow \vec{a}$ lin. neodvisen od $\vec{b}$.

$(D\Phi)(t,s) = \begin{array}{c} \\ \Phi_t \end{array} \begin{bmatrix} x_t & x_s \\ y_t & y_s \\ z_t & z_s \end{bmatrix} \Phi_s$ $\qquad (x_t, y_t, z_t) \times (x_s, y_s, z_s) \neq 0$

$\Updownarrow$

Kdaj je rang 2? $\Leftrightarrow$ $\qquad$ rang $= 2$.

Implicitno.

·) $\forall a \in M \ \exists U \ni a$, $F_1,\dots,F_m \in C^1(U)$

i) $\operatorname{rang}(F_1,\dots,F_m) = m$

ii) $M \cap U = \{x \in U;\ F_i(x) = 0 \ \forall i\}$

Eksplicitno (kot graf): Po permutaciji koordinat: $\forall a \in M \ \exists U \ni a$, da je

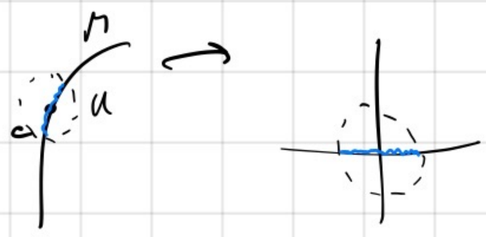
$M \cap U = \{(x_1,\dots,x_n, \varphi_1(x_1,\dots,x_n), \dots, \varphi_m(x_1,\dots,x_n)),\ (x_1,\dots,x_n) - D^{odp} \subseteq \mathbb{R}^n\}$.

ali $(x', \varphi'(x))$

Parametrizacija: $\forall a \in M \ \exists U \ni a$. $D \subseteq \mathbb{R}^n$: $\Phi: D \to \mathbb{R}^{n+m}$, rang $\Phi = n$.

$\Phi(D) \subseteq M \cap U$

Trditev: Naj bo M v $\mathbb{R}^{n+m}$ podmnogoterost dimenzije n. Za $\forall a \in M$ obstaja taka odprta okolica U točke a, $U \subseteq \mathbb{R}^{n+m}$ in tak C^1 difeomorfizem $\Phi: U \to W \subseteq \mathbb{R}^{n+m}$, da je $\Phi(M \cap U)$ oblike $\Phi(M \cap U) = D \times \{0\}^n$ kjer je D odprta množica v $\mathbb{R}^n$.



Opomba: Podmnogoterost lahko lokalno izravnamo (preko difeomorfizma)

Dokaz: Lokalno je $M \cap U = \{(x, \varphi'(x)),\ x' \in D^{odp} \subseteq \mathbb{R}^n,\ \varphi \in C^1(D),\ \varphi: D \to \mathbb{R}^m\}$.

$\Phi: U \to W$

$\Phi(\underbrace{x_1,\dots,x_n}_{x'}, x_{n+1},\dots,x_{n+m}) = (x_1,\dots,x_n, x_{n+1} - \varphi_1(x'), \dots, x_{n+m} - \varphi_m(x'))$

Očitno: $\Phi(M\cap U) = D\times\{0\}^m$.

$\Phi^{-1}(x_1,\dots,x_n,y_1,\dots,y_m) = (x_1,\dots,x_n, y_1+\varphi_1(x'),\dots,y_m+\varphi_m(x'))$.

Odvod Φ: $(D\Phi)(x) = \begin{bmatrix} I & 0 \\ (-D\varphi)(x') & I \end{bmatrix}$ je očitno obrnljiva

$\Rightarrow \Phi$ je difeomorfizem. ■

Opombe: ·) $M\cap U$, $F_1=\dots=F_m=0$ in rang$(\overbrace{F_1,\dots,F_m}^{F}) = m$: $(DF)(a) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_{n+m}} \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_{n+m}} \end{bmatrix}$ (a)

$\Psi(x) = (F_1(x),\dots,F_m(x))$

so vrstice lin. neodvisne, zato jih lahko dopolnimo do baze po vrsticah, dobimo obrnljiv odvod, po izreku o inverzu, dobimo difeomorfizem.

·) Trditev velja tudi v nasprotno smer. $\Phi(M\cap U) = D\times\{0\}^m$. Definicijska bodo vsaj m funkcij Φ, dobimo difeomorfizem, odvod polnega ranga.

$x^2+y^2=1$
$(0,1)$

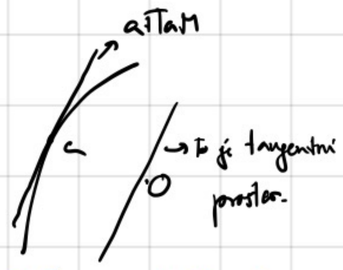
Tangentni prostor: $\gamma: (\alpha,\beta)\subseteq\mathbb{R}\to\mathbb{R}^n$, $\gamma\in C^1$
$t\mapsto (x_1(t),\dots,x_n(t))$

$\dot\gamma(t) = (\dot x_1(t),\dots,\dot x_n(t))$

Naj bo $M\subseteq\mathbb{R}^n$ podmnogoterost in $a\in M$. Tangentni prostor na M v točki a je
$T_aM = \{\dot\gamma(t_0);\ \gamma:(\alpha,\beta)\subseteq\mathbb{R}\to M\ C^1$ preslikava $t_0\in(\alpha,\beta), \gamma(t_0)=a\}$ Vse krivulje skozi a, tangentni vektorji teh krivulj.

Eksplicitno ???

Trditev: Naj bo $M\subseteq\mathbb{R}^n$ podmnogoterost in $a\in M$. Potem je T_aM vektorski podprostor dimenzije, enake kot mnogoterost M.
$\dim M = \dim T_aM$. $(\forall a\in M)$.



Preklik, saj mora imeti ničlo.

Dokaz: Lokalno je M graf nad enim od koordinatnih podprostorov (s permutacijo koordinat - linearna preslikava)
$\dim M = d$
$a\in M$. $\exists U\ni a$, da je $M\cap U = \{\underbrace{x_1,\dots,x_d}_{x'}, \overbrace{\varphi_1(x_1,\dots x_d),\dots,\varphi_{n-d}(x_1,\dots,x_n)}^{\varphi(x')};\ x'\in D^{odp}\subseteq\mathbb{R}^d\}$. → lokalno je graf
$\gamma:(\alpha,\beta)\subseteq\mathbb{R}\to M$.
$\gamma(t_0) = (a', \varphi(a'))\in M$.
$\forall t\in(\alpha,\beta)$ je $\gamma(t)\in M \Rightarrow \gamma(t)$ je oblike $(x'(t), \varphi(x'(t)))$
Odvod: $\dot\gamma = (\dot x'(t), (D\varphi)(x'(t))\dot x'(t))$
$= \underbrace{\begin{bmatrix} I \\ D\varphi \end{bmatrix}}_{d} \dot x'(t)$ v t

i) $v_0\in\mathbb{R}^d$ $\quad t\mapsto (a', a'+tv^0) = x'(t)$
$\dot x' = v_0$
$\Rightarrow T_aM = \operatorname{Im}\begin{bmatrix} I \\ D\varphi(a') \end{bmatrix}$
$\Rightarrow T_aM$ je vektorski podprostor, dimenzije d saj $(\operatorname{rang}\begin{bmatrix} I \\ D\varphi(a') \end{bmatrix}.)$ ■

Posledica: Če je M lokalno podana kot graf: $M\cap U = \{(x',\varphi(x')),\ x'\in D\subset\mathbb{R}^d,\ \varphi: D\xrightarrow{C^1}\mathbb{R}^{n-d}\}$
$a = (a', \varphi(a'))\in M\cap U$, $a'\in D$
Potem je: $T_aM = \operatorname{Im}\begin{bmatrix} I \\ D\varphi(a') \end{bmatrix}$ (graf odvoda $D\varphi(a')$)

Implicitno podana

Trditev: Naj bo mnogoterost M v okolici $U\ni a(\in M)$ podana implicitno kot $M\cap U = \{x\in U, F_1=\dots=F_m=0\}$ rang $(F_1,\dots,F_m) = n-d$
Tedaj je $T_aM = \ker((DF)(a))$

Opomba: v bo tangentni vektor $\Leftrightarrow (DF)(v)=0$

$$\begin{array}{c} \\ n-d \end{array}\overset{n}{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ & & \end{bmatrix}} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = 0,\ \nabla F_1(a), \nabla F_2(a),\dots,\nabla F_{n-d}(a)$$ so paralelni.

Ti vektorji razpenjajo ortogonalni komplement tangentnega prostora.
$\mathcal{L}(\nabla F_1(a),\dots,\nabla F_{n-d}(a)) = (T_aM)^\perp$.

Dokaz: Naj bo $\gamma : (\alpha,\beta) \subseteq \mathbb{R} \xrightarrow{C^1} M$

$\gamma(t_0) = a$.

$F = (F_1, \dots, F_{n-d})$

$\Rightarrow F \circ \gamma = 0 \quad \forall t \in (\alpha,\beta)$

Odvajamo: $(DF)(\gamma(t))\,\dot\gamma(t) = 0$

$t := t_0 \Rightarrow (DF)(a)\,\dot\gamma(t_0) = 0$

$\Rightarrow \dot\gamma(t_0) \in \ker((DF)(a))$

↳ vsi tangenti.

$\Rightarrow T_aM \subseteq \ker((DF)(a))$ (oba dimenzije d)

Enakost dobimo iz enakosti dimenzij.

Parametrično podana

Trditev: Naj bo $M \subseteq \mathbb{R}^n$ podmnogoterost dimenzije d. Naj bo $a \in M$ in naj bo $\Phi : D^{\text{odp}} \subseteq \mathbb{R}^d \xrightarrow{C^1} \mathbb{R}^n$ preslikava ranga d in taka, da je:

i) $\Phi(t_0) = a$ za neki $t_0 \in D$

ii) $\Phi(D) = M \cap U$ za neko okolico U točke $a \in \mathbb{R}^n$.

Tedaj je $T_aM = \operatorname{Im}((D\Phi)(t_0))$

Dokaz: Naj bodo $F_1, \dots, F_{n-d} \in C^1(U)$, $a \in U$, rang$(F_1, \dots, F_{n-d})$ lokalne definicijske funkcije za $M \cap U$.

Naj bo še $F = (F_1, \dots, F_{n-d})$

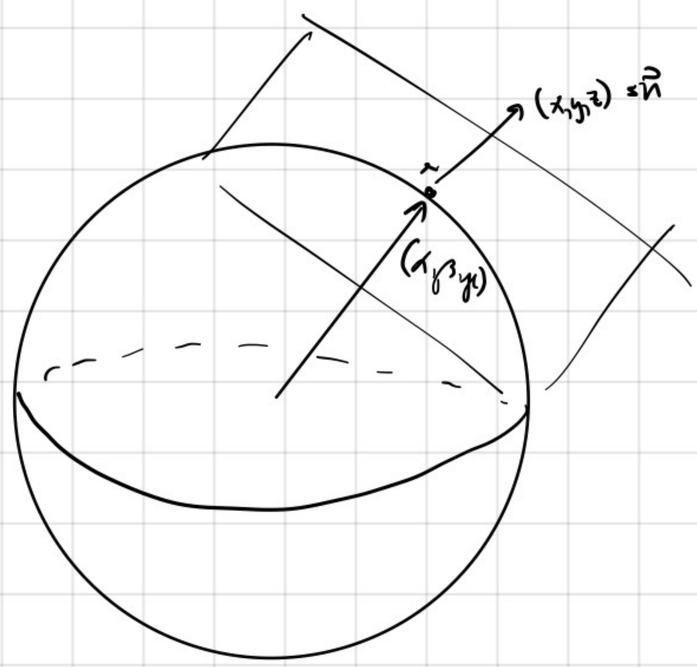
Tedaj: $F \circ \Phi = 0$

Odvajamo: $(DF)(\Phi(t))\,(D\Phi)(t) = 0$

$t = t_0$: $(DF)(a)\,(D\Phi)(t_0) = 0$

$\Rightarrow \operatorname{Im}(D\Phi)(t_0) \subseteq \ker(DF)(a)$ (Oba dimenzije d)

Enakost sledi iz enakosti dimenzije.



Zgledi: i) $M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$a = (\alpha,\beta,\gamma) \in M$

$F(x,y,z) = x^2+y^2+z^2-1$

$(DF)(x,y,z) = [2x \;\; 2y \;\; 2z]$

$(DF)(\alpha,\beta,\gamma) = [2\alpha \;\; 2\beta \;\; 2\gamma]$

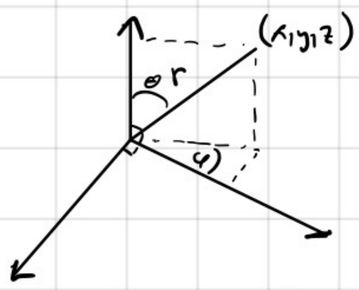
$T_{(\alpha,\beta,\gamma)}M = \{(X,Y,Z) ;\ \alpha X + \beta Y + \gamma Z = 0\}$

To je tangentni prostor na M v (α,β,γ).

afino premaknjeno

Tangentna ravnina: $\{(x,y,z) ;\ \alpha x + \beta y + \gamma z = 1\} = a + T_aM$

ii)



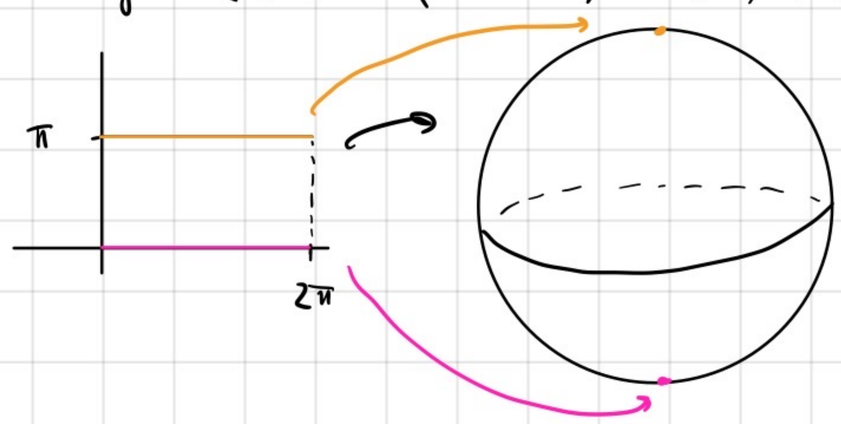
$r = \sqrt{x^2+y^2+z^2}$ $\qquad x = r\sin\theta\cos\varphi$

$\varphi \in [0, 2\pi)$ $\qquad y = r\sin\theta\sin\varphi$

$\theta \in [0, \pi]$ $\qquad z = r\cos\theta$

$S^2(0,a) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2\}$

Parametrizacija: $(\varphi,\theta) \longmapsto (a\cos\varphi\sin\theta,\ a\sin\varphi\sin\theta,\ a\cos\theta)$; $\varphi \in [0,2\pi)$, $\theta \in [0,\pi]$



Regularna? $\vec{r}_\varphi, \vec{r}_\theta$

$\vec{r}_\varphi \times \vec{r}_\theta \neq 0$.

$\vec{r}_\varphi = (-a\sin\varphi\sin\theta,\ a\cos\varphi\cos\theta,\ 0)$

$\vec{r}_\theta = (a\cos\varphi\cos\theta,\ a\sin\varphi\cos\theta,\ -a\sin\theta)$

$\vec{r}_\varphi \times \vec{r}_\theta = \left(-a^2\cos\varphi\sin^2\theta,\ -a^2\cos^2\theta\sin\varphi,\ -a^2\sin\theta\cos\theta\right) = -a^2\sin\theta\,(\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta)$

$|\vec{r}_\varphi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\theta + a^4\sin^2\theta\cos^2\theta} = a^2\sqrt{\sin^2\theta} = a^2\sin\theta \neq 0$ za $\theta \in (0,\pi)$. → tukaj je dobra parametrizacija

Splošno: $\vec{r}(s,t)$, normala bo $\vec{r}_s \times \vec{r}_t = \vec{n}$.

Zgled: $\Gamma = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ x+y+z=0\}$.

$F(x,y,z) = x^2+y^2+z^2-1$

$G(x,y,z) = x+y+z$

Odvod: $\begin{bmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ $\qquad xX+yY+zZ=0$ in $X+Y+Z=0$

$a = (\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, -\frac{2}{\sqrt6})$ ustrezna obenem pozitivna. $a \in \Gamma$

$T_a\Gamma = \{(X,Y,Z) \in \mathbb{R}^3 \mid \frac{1}{\sqrt6}X + \frac{1}{\sqrt6}Y - \frac{2}{\sqrt6}Z = 0;\ X+Y+Z=0\}$

$\{$ -"- $\mid X+Y-2Z=0;\ X+Y+Z=0\} = \{\lambda(1,-1,0);\ \lambda\in\mathbb{R}\}$ Tangentni prostor.

$\Rightarrow Z=0,\ X+Y=0$

Tangentna premica / tangenta na $T_a\Gamma = \{(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, -\frac{2}{\sqrt6}) + \lambda(1,-1,0);\ \lambda\in\mathbb{R}\}$

$x = \frac{1}{\sqrt6}+\lambda \qquad \frac{x-\frac{1}{\sqrt6}}{1} = \frac{y-\frac{1}{\sqrt6}}{-1}$

$y = \frac{1}{\sqrt6}-\lambda$

$z = -\frac{2}{\sqrt6} \qquad z = \frac{-2}{\sqrt6}$.

Zgled: $M = \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\} \subseteq \mathbb{R}^{n\times n}$

$F(X) = \det X - 1$, podmnogotenost dimenzije n^2-1.

Kolikšen je $T_A M$?

$n=2$: $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad-bc = 1$

$F(X) = xw - yz - 1$

$(DF)(X) = \begin{bmatrix} w & -z & -y & x \end{bmatrix}$

$(DF)(A) = \begin{bmatrix} d & -c & -b & a \end{bmatrix}$

$T_A M = \{\begin{bmatrix} X & Y \\ Z & W \end{bmatrix};\ dX - cY - bZ + aW = 0\}$

Splošen n: $F(X) = \det X - 1$; $X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$

Priročno: $(\tilde{A})_{ij} = (-1)^{i+j} A_{ji}$ $\qquad A\tilde{A} = \det A \cdot I$ $\quad \det A = 1$

$\frac{\partial F}{\partial x_{ij}}(A) = (-1)^{i+j} A_{ij} = \det(A_{ij})$

$X \in \mathbb{R}^{n\times n} \in T_A M \Leftrightarrow \sum_i \sum_j (-1)^{i+j} A_{ij} X_{ij} = 0 = \sum_j \sum_i \tilde{A}_{ji} X_{ij} = \mathrm{Sl}(\tilde{A}\cdot X) = \mathrm{Sl}(A^{-1}X)$

$T_A M = \{X \in \mathbb{R}^{n\times n} \mid \mathrm{Sl}(A^{-1}X) = 0\}$

Krivulje v $\mathbb{R}^3$ (ali $\mathbb{R}^2$).

$\Gamma \subseteq \mathbb{R}^3$, $\Gamma \neq \emptyset$.

$\dim \Gamma = 1$

i) $a \in \Gamma$ implicitno $\exists U$ okolica in $F, G \in C^1(U)$, da je $\Gamma \cap U = \{(x,y,z) \in U \mid F(x,y,z)=0,\ G(x,y,z)=0\}$.

$\operatorname{rang}\begin{bmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{bmatrix} = 2$

$(a,b,c) \in \Gamma$ eksplicitno.

ii) $\Gamma \cap U$ je graf nad eno od osi: x-os, y-os, z-os.

Nad x-osjo: $\Gamma \cap U = \{(x, \varphi(x), \psi(x));\ \varphi, \psi$: neki $I \subseteq \mathbb{R}$ okolici $\to \mathbb{R}\}$

Ostalo podobno.

iii) Parametrično: $t \mapsto (x(t), y(t), z(t)) = \vec{r}(t)$

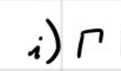
$\dot{\vec{r}}(t) \neq 0 \ \forall t$. Taka je regularna parametrizacija.

$\hookrightarrow \dot{x}(t) \neq 0$ ali $\dot{y}(t) \neq 0$ ali $\dot{z}(t) \neq 0$.

Vsako krivulja lahko parametriziramo globalno. Denimo, da je Γ povezana: iz enega kosa

i) Γ kompaktna, obstaja $\vec{r} : S^1 \to \Gamma$

ii) Γ ni kompaktna, $\vec{r} : \mathbb{R} \to \Gamma$ ali $(0,1) \to \Gamma$

iii)  $\subseteq \mathbb{R}^3$.

$\Gamma = \{(\lambda, x_0) ; x \in [0,1]\}$ ni mnogoterost, saj vsebuje svoje robove.

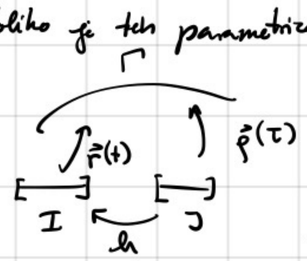
$\partial\Gamma = \{(1,0,0), (0,0,0)\}$ rob podmnogoterosti Imenujemo jo podmnogoterost z robom.

iv) $\vec{r} : [0,1] \to \Gamma$

v) $\vec{r} : [0,1) \to \Gamma$ ali $[0,\infty) \to \Gamma$

$\vec{r} : I \subseteq \mathbb{R} \to \Gamma$ reg. parametrizacija, $\dot{\vec{r}} \neq 0$ in bijektivna

Koliko je tem parametrizacij?



Vsaka zase sta regularni parametrizaciji

Lahko tvorimo $h = \vec{r}^{-1} \circ \vec{\rho}$ oz. $\vec{\rho} = \vec{r} \circ h$

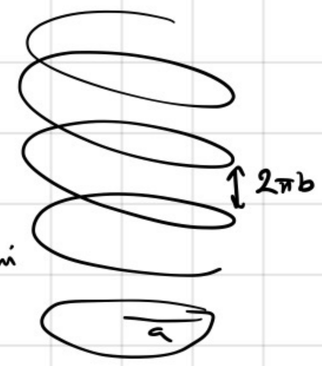
$h : J \to I$ je difeomorfizem.

Odvod: $\underset{\neq 0}{\rho'} = \underset{\neq 0}{\dot{\vec{r}}(h(t))} h'(t) \Rightarrow h'(t) \neq 0$

Vijačnica: $a > 0$

$b \neq 0$

$t \mapsto (\underbrace{a\cos t, a\sin t}_{\text{krožnica v xy ravnini}}, bt)$ $\vec{r}(t)$



$a = (a, 0, 0)$

$\dot{r}(t) = (-a\sin t, a\cos t, b) \neq 0$, zato regularna parametrizacija $\mathbb{R} \to \Gamma$.

$T_{(a,0,0)}\Gamma = \{\lambda(0, a, b) ; \lambda \in \mathbb{R}\}$.

Tangenta: $\{(a, \lambda a, \lambda b) ; \lambda \in \mathbb{R}\}$

Ploskve v $\mathbb{R}^3$

$\Sigma \neq 0$, $\dim \Sigma = 2$

·) $\forall (a,b,c) \in \Sigma \ \exists U$ okolica $F \in C^1(U)$ $[F_x, F_y, F_z] \neq 0$.

$\Sigma \cap U = \{(x,y,z) \in U \mid F(x,y,z) = 0\}$. $(DF)(a,b,c) \neq 0$

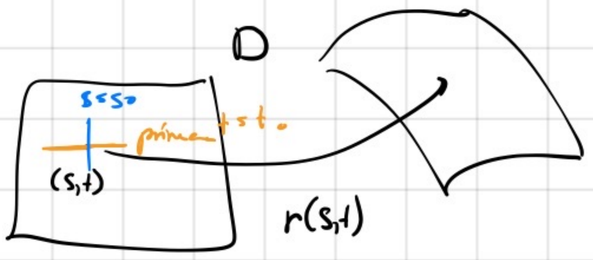
··) Če $F_z = (a,b,c) \neq 0$ lokalno je Σ graf nad (x,y) ravnino. $\Sigma \cap U = \{(x,y, z(x,y)) \mid (x,y) \in D^{\text{odp}} \subseteq \mathbb{R}^2, (a,b) \in D^{\text{to}} \subseteq \mathbb{R}^2\}$

···) $\Phi : D \subseteq \mathbb{R}^2 \to \Sigma$

$\Phi(s_0, t_0) \to (a,b,c)$

$\operatorname{rang}(D\Phi)(s_0, t_0) = 2$

$\Phi(s,t) = r(s,t) = (x(s,t), y(s,t), z(s,t))$



$s \mapsto r(s, t_0)$
$t \mapsto r(s_0, t)$ } koordinatni krivulji

(s,t) sta koordinati na Σ

Pogoj na odvod: $\operatorname{rang} \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = 2$. $\det \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} \neq 0$

$(s,t) \to (x(s,t), y(s,t))$ ima inverz

dobimo inverz.

$\begin{bmatrix} r_s & r_t \end{bmatrix}$

$r_s \times r_t \neq 0$ tudi da rang $= 2$. r_s in r_t sta tangentna vektorja

$\vec{r}_s(s_0, t_0)$ na prvo koordinatno krivulji

$\vec{r}_t(s_0, t_0)$ na drugo koordinatno krivulji

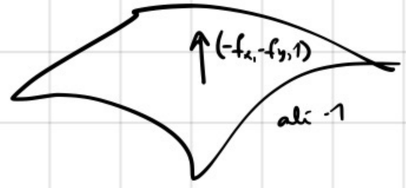
Normala na tangentni prostor je $\vec{r}_s \times \vec{r}_t$.

Enačba tangentne ravnine: $(\vec{R} - \vec{r}(s,t))(\vec{r}_s \times \vec{r}_t) = 0$

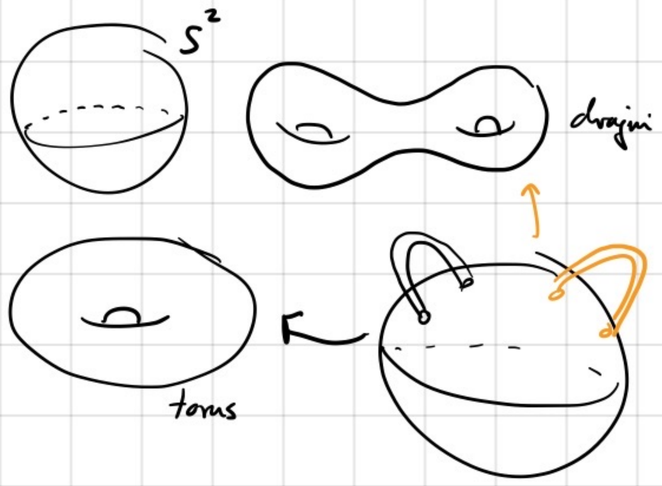
↳ točka na ravnini

Poseben primer: Če imamo graf: $(x,y) \mapsto (x,y,f(x,y)) = \vec{r}(x,y)$

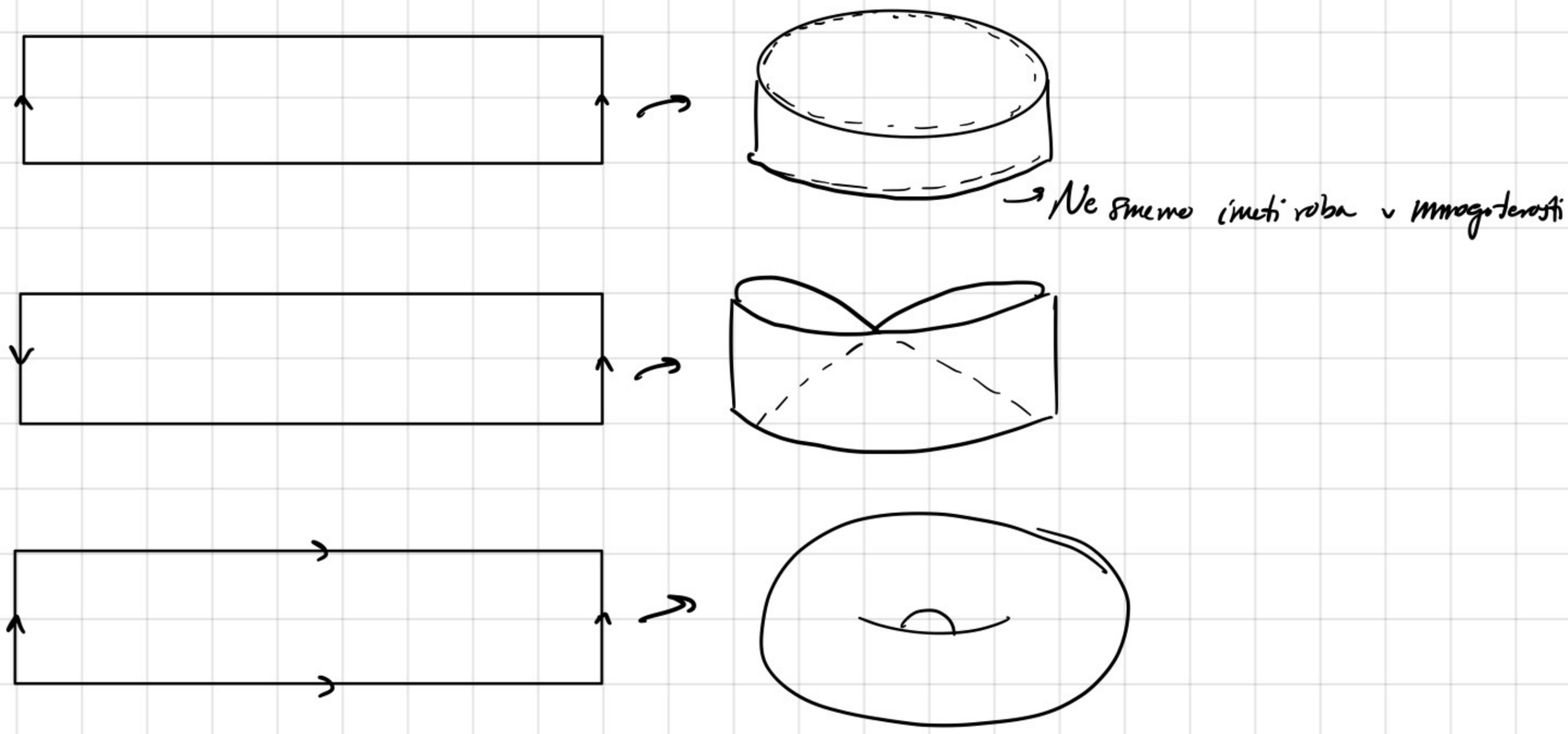
$\vec{r}_x(1,0,f_x)$, $\vec{r}_y(0,1,f_y)$ $\Big\} \to \vec{n} = (-f_x, -f_y, 1)$



Ploskve:



Kompaktne in ploskve brez roba, sklenjene



Všečnji Möbiusov trak, ne moremo realizirati v $\mathbb{R}^3$. Zakaj?

Zgled: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$r(\varphi,\theta) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ $\qquad \vec{n} = \vec{r_\varphi} \times \vec{r_\theta}$

$\theta = \frac{\pi}{4}$ ali $45°$

$\varphi = 15°$ ali $\frac{\pi}{12}$

Enačba tangentne ravnine v $r(\frac{\pi}{12}, \frac{\pi}{4}) = (\cos\frac{\pi}{12}\cdot\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\sin\frac{\pi}{12}, \frac{\sqrt{2}}{2})$

$\frac{1}{2} = \sin\frac{\pi}{6} = \sin(2\cdot\frac{\pi}{12}) = 2\cdot\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

$\left.\begin{array}{l}\sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12} = \frac{1}{4} \\ \sin^2\frac{\pi}{12} + \cos^2\frac{\pi}{12} = 1\end{array}\right\}$ $\sin^2\frac{\pi}{12}\cos^2\frac{\pi}{12} = \frac{1}{16}$ $\quad$ (Kvadratna za $u = \sin^2\frac{2\pi}{12}$)

$u^2 - u + \frac{1}{16} = 0 \qquad u_{1,2} = \frac{1 \pm \sqrt{1-\frac{4}{4}}}{2} = \frac{1 \pm \frac{\sqrt{3}}{2}}{2} = \frac{2 \pm \sqrt{3}}{4} = \frac{4 \pm 2\sqrt{3}}{2\cdot 4} = \frac{(\sqrt{3}\pm 1)^2}{(2\sqrt{2})^2}$

$\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Točka: $\left(\frac{\sqrt{3}+1}{4}, \frac{\sqrt{3}-1}{4}, \frac{\sqrt{2}}{2}\right)$

$\vec{r_\varphi} = (-\sin\varphi\sin\theta, \cos\varphi\sin\theta, 0)$

$\vec{r_\theta} = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)$

$\vec{n} = \vec{r_\varphi} \times \vec{r_\theta} = (-\sin^2\theta\cos\varphi, -\sin\varphi\sin^2\theta, -\sin\theta\cos\theta)$

$\varphi = \frac{\pi}{12}, \theta = \frac{\pi}{4} \Rightarrow \vec{n} = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\cdot\frac{1}{2}, \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{1}{2}, \frac{1}{2}\right)$

$$\frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}+1}{2\sqrt{2}}y + z = \frac{\sqrt{3}+1}{2\sqrt{2}}\cdot\frac{\sqrt{3}+1}{4} + \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{3}-1}{4} + \frac{\sqrt{2}}{2}$$

$$= \frac{4+2\sqrt{3}}{8\sqrt{2}} + \frac{4-2\sqrt{3}}{8\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

Taylorjeva formula:

$h = (h_1, \dots, h_n) \in \mathbb{R}^n$

$D_h = h_1 \frac{\partial}{\partial x_1} + \dots + h_n \frac{\partial}{\partial x_n}$ operator na funkcije v smeri vektorja h.

$f \mapsto D_h f = h_1 \frac{\partial f}{\partial x_1} + \dots + h_n \frac{\partial f}{\partial x_n} = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}$

$$(D_h^2 f) = D_h(D_h f) = D_h\left(\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}\right) = \sum_{k=1}^{n} h_k \left(D_h \frac{\partial f}{\partial x_k}\right) = \sum_{j=1}^{n} h_j \left(\sum_{i=1}^{n} h_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$n=2$: $D_{(\alpha,\beta)} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$

$D_{(\alpha,\beta)}^2 = \alpha^2 \frac{\partial^2}{\partial x^2} + 2\alpha\beta \frac{\partial^2}{\partial x \partial y} + \beta^2 \frac{\partial^2}{\partial y^2}$

$$D_{(\alpha,\beta)}^m = \sum_{j=0}^{m} \binom{m}{j} \frac{\partial^m}{\partial x^j \partial y^{j-m}}$$

Izrek: Naj bo $D^{\text{odp}} \subseteq \mathbb{R}^n$ in $f \in C^{k+1}(D)$. Naj bo $a \in D$ in $h \in \mathbb{R}^n$ tak vektor, da $a \to a+th$ daljica med a in $a+th$ leži v D. Tedaj obstaja tak $\theta \in (0,1)$, da je $f(a+h) = f(a) + \frac{1}{1!}(D_h f)(a) + \frac{1}{2!}(D_h^2 f)(a) + \dots + \frac{1}{k!}(D_h^k f)(a) + R_k$, kjer je $R_k = \frac{1}{(k+1)!}(D_h^{k+1} f)(a+\theta h)$

polinom v h stopnje $\nearrow$ največ k.

Opomba: ·) $h \mapsto f(a) + \frac{1}{1!}(D_h f)(a) + \dots + \frac{1}{k!}(D_h^k f)(a)$ $\overset{\text{s}}{=} \sum_{0}^{k} \frac{1}{j!}(D_h^j f)(a)$ je polinom stopnje $\leq k$.

··) Če je $f \in C^\infty(D)$, potem lahko tvorimo Taylorjevo vrsto: $\sum_{0}^{\infty} \frac{1}{j!}(D_h^j f)(a)$, $(e^{-\frac{1}{x}}$ npr.$)$

Gotovo konvergira za $h=0$. Tudi če konvergira za neki $h \neq 0$, ne konvergira k funkciji $f(a+h)$.

···) Če Taylorjeva vrsta konvergira k $f(a+h)$ za vse $|h| < \delta$, potem $f(a+h) = \sum_{0}^{\infty} \frac{1}{j!}(D_h^j f)(a)$ in je f v okolici a realno analitična.

Dokaz: $\varphi(t) = f(a+th) = f(a_1+th_1, a_2+th_2, \dots, a_n+th_n)$

$\varphi(0) = f(a)$

$\varphi'(t) = \frac{\partial f}{\partial x_1}(a+th) \cdot h_1 + \frac{\partial f}{\partial x_2}(a+th) h_2 + \dots + \frac{\partial f}{\partial x_n}(a+ht) \cdot h_n$

$\varphi'(0) = (D_h f)(a)$

$\varphi''(t) = (D_h^2 f)(a+th)$

$\vdots$

$\varphi^{(j)}(t) = (D_h^j f)(a+th)$

Taylorjeva formula za $\varphi(1) = \varphi(0) + \frac{1}{1!}\varphi'(0) + \frac{1}{2!}\varphi''(0) + \dots + \frac{1}{k!}\varphi^{(k)}(0) + \frac{1}{(k+1)!}\varphi^{(k+1)}(\theta)$; $\theta \in (0,1)$.

Za $t=0$; $a+th = a$, $t=1$, $a+th = a+h$

$$\Rightarrow \varphi(1) = f(a+h) = f(a) + \frac{1}{1!}(D_h f)(a) + \dots + \frac{1}{k!}(D_h^k f)(a) + \underbrace{\frac{1}{(k+1)!}(D_h^{k+1} f)(a+\theta h)}_{R_k}$$

■

Zgled: $f(x,y) = e^{xy}$ $\overset{\text{s}}{=} \sum_{0}^{\infty} \frac{x^n y^n}{n!}$ $\forall x,y$, $(a,b) := (0,0)$

$f(x,y) = f(0,0) + \frac{1}{1!}(D_{x,y} f)(0,0) + \dots + \frac{1}{k!}(D_{x,y}^k f)(0,0)$

$h = (x,y)$

$$(D_{x,y}^n f)(0,0) = \Big(\underbrace{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}}_{D_{x,y}}\Big)^n f\Big)(0,0) = \Big(\sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \frac{\partial^n}{\partial x^j \partial y^{n-j}} f\Big)(0,0)$$

$\left(\frac{\partial^j}{\partial x^j} f\right) = y^j e^{xy}$ $\quad n-j = j$, sicer dobimo 0.

$n = 2j$

$$\left(\frac{\partial^{2j}}{\partial x^j \partial y^j} f\right)(0,0) = j! \cdot 1$$

$(D_{x,y}^n f)(0,0) \neq 0$ samo za sode n.

$(D_{x,y}^{2j} f)(0,0) = j! \binom{2j}{j} = \frac{(2j)!}{j!j!} j! = \frac{(2j)!}{j!}$

$\frac{1}{(2j)!}(D_{x,y}^{2j} f)(0,0) = \frac{1}{j!}$

$$\Rightarrow \sum_{0}^{\infty} \frac{1}{j!} x^j y^j$$

Posledica: Naj bodo D, a, f, h kot v izreku. Potem je:

i) $R_k = o(|h|^k)$ ko gre $h \to 0$

ii) $R_k = O(|h|^{k+1})$ ko gre $h \to 0$.

Opomba: ·) $R_k = o(|h|^k) \Leftrightarrow \lim_{h\to 0} \frac{|R_k|}{|h|^k} = 0$

·) $\frac{|R_k|}{|h|^{k+1}}$ je omejen: $\exists M<\infty$, da je $\frac{|R_k|}{|h|^{k+1}} < M$ za $|h|<\delta$

Dokaz: $R_k = \frac{1}{(k+1)!}(D_h^{k+1} f)(a+\theta h)$ ostanek je oblike $\qquad D_{j_1} = \frac{\partial f}{\partial x_{j_1}} \quad D_{j_2} = \frac{\partial f}{\partial x_{j_2}}$

$$(D_h^{k+1} f)(a+\theta h) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\dots\sum_{j_{k+1}=1}^{n}\underbrace{(D_{j_1}D_{j_2}\cdots D_{j_{k+1}} f)}_{\text{odvodi po spremenljivkah } x_{j_1},\dots,x_{j_{k+1}}}(a+\theta h)(\underbrace{h_{j_1}\cdots h_{j_{k+1}}}_{\text{zmnožek}})$$

Ker je $f \in C^{k+1}(D)$ so vsi njeni $k+1$ odvodi zvezni in so na daljici med a in $a+h$ enakomerno omejeni

$|(D_{j_1}\dots D_{j_{k+1}}) f(a+\theta h)| \le M$

$|h_{j_1}\dots h_{j_{k+1}}| \le |h|\cdot\dots\cdot|h| = |h|^{k+1}$

$\Rightarrow |R_k| \le C|h|^{k+1}$

?? $\quad f(x,y) \quad a=(0,0)$

$$\frac{1}{(k+1)!}(D_{(x,y)}^{n} f)(0,0) = \frac{1}{(n+m)!}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)^{n+m}(0,0)$$

$$\sum \binom{n+m}{j} x^j y^{n+m-j} \frac{\partial^{n+m}}{\partial x^j \partial y^{n+m-j}}(0,0)$$

Zanima nas $x^n y^n$: $\frac{1}{(n+m)!}\binom{n+m}{n} x^n y^m \frac{\partial^{n+m}}{\partial x^n \partial y^m}$

$$\Rightarrow \underbrace{\frac{1}{n!\,m!}\left(\frac{\partial^{n+m}}{\partial x^n \partial y^m} f\right)(0,0)}_{\text{koeficient za potenco.}} x^n y^n$$

$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$

$f^n(0) = a_n \cdot n!$

drugi način.

$f(x,y) = \sum_n \sum_m a_{nm} x^n y^n$

$\frac{\partial^{n+m} f}{\partial x^n \partial y^n}(0,0) = a_{nm}\, n!\, m!$

Ekstremi funkcij več spremenljivk:

$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in D$.

·) funkcija ima v a lokalni maksimum, če obstaja tak $r>0$, da je $f(a) \ge f(x)$ $\forall x \in D \cap K(a,r)$.

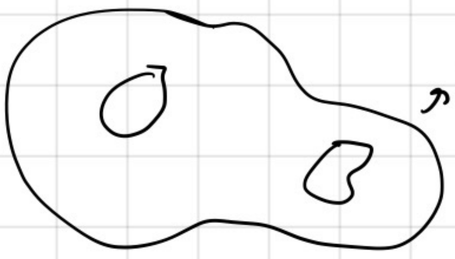
Če velja: $f(a) > f(x)$ $(\forall x \in D \cap K(a,r)) \setminus \{a\}$, potem ima f v a strogi lokalni maksimum.

·) f ima v a maksimum (globalni glede na D), če je $f(a) \ge f(x)$ $\forall x \in D$.

Če je $f(a) > f(x)$ $\forall x \in D \setminus \{a\}$, ima f v a strogi maksimum.

Podobne definicije veljajo za lokalne minimume oz. globalni minimum.

Opomba: Če je K kompaktna podmnožica $\mathbb{R}^n$ in $f: K \to \mathbb{R}$ zvezna ima f globalne ekstreme na K.



notranjost in rob.

Definicija: Naj bo D odprta $\subseteq \mathbb{R}^n$ in f vsaj $C^1(D)$ (dvakrat diferencirabilna). $a \in D$ je stacionarna/kritična točka za f, če je $(Df)(a) = 0$ oz. $\frac{\partial f}{\partial x_1}(a) = \dots = \frac{\partial f}{\partial x_n}(a) = 0$.

Trditev: Naj bo $a \in D^{odp} \subseteq \mathbb{R}^n$, $f \in C^1(D)$. Če ima f v a lokalni ekstrem, je a kritična točka za f.

oz. $\frac{\partial f}{\partial x_1}(a) = \dots = \frac{\partial f}{\partial x_n}(a) = 0$ oz. $(Df)(a)$

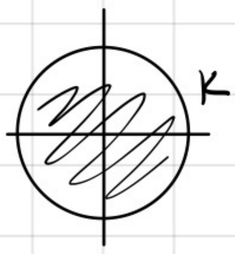
Dokaz: Poglejmo si funkcijo $\varphi_1 : t \mapsto f(t, a_2, \dots, a_n)$ v okolici a_1.

Če ima f v a lokalni ekstrem, ima φ_1 v a_1 lokalni ekstrem, zato je $\varphi_1'(a_1) = 0 = \frac{\partial f}{\partial x_1}(a)$.

Podobno za ostale komponente. ■

Zgled: $K = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 3^2\}$

$f(x,y) = x^2 - xy + y^2 - 3x + 4$



Poiščemo globalni minimum in maksimum.

1.) Poiščemo stacionarne točke za K.

$$\left.\begin{aligned} f_x &= 2x - y - 3 = 0 \\ f_y &= -x + 2y = 0 \end{aligned}\right\} \Rightarrow 4y - y = 3 \Rightarrow \underline{y = 1} \Rightarrow \underline{x = 2} \quad (2,1) \in K.$$

$f(2,1) = 1$ Kandidat za ekstrem. $\Rightarrow$ globalni minimum

2.) Preveriti moramo še rob:

$t \mapsto (3\cos t, 3\sin t)$; $t \in (0, 2\pi]$.

$f(3\cos t, 3\sin t) = g(t) = 13 - 9\cos t - 9\sin t \cos t$

$g'(t) = -9\overset{1-\sin^2 t}{\cos^2 t} + 9\sin^2 t + 9\sin t = 0$

$= -1 + \sin^2 t + \sin^2 t + \sin t = 0$

$= 2\sin^2 t + \sin t - 1 = 0$

$(\sin t)_{1,2} = \frac{-1 \pm \sqrt{8+1}}{4} = \frac{-1 \pm 3}{4}$

i) $\sin t = -1 \Rightarrow t = \frac{3\pi}{2}, T(0,-3)$ $\quad f(0,-3) = 13$

ii) $\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}, T\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ $\quad f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 13 - \frac{27\sqrt{3}}{4}$

$\Rightarrow t = \frac{5\pi}{6}, T\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ $\quad f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 13 + \frac{27\sqrt{3}}{4}$ $\Rightarrow$ globalni maksimum

$t = 0$ $\quad T(3,0)$

$t = 2\pi$ $\quad T(3,0)$ $\quad f(3,0) = 4$

Radi bi še način za določitev min/maks.

Potrebni in zadostni pogoji, da je kritična točka lokalni ekstrem (na drugo odvode):

$f \in C^2(D)$, $D^{odp} \subseteq \mathbb{R}^n$.

$$(Hf)(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}(x)$$

Če je $f \in C^2$, so mešani odvodi enaki, potem je $(Hf)^T = (Hf)$, je simetrična. (ima realne lastne vrednosti v vsaki točki), da se jih diagonalizirati.

(Hf) - Hessejna matrika.

$(Hf) \geq 0$: pozitivno semidefinitna $\quad \langle (Hf)v, v \rangle \geq 0 \ \forall v \iff$ vse lastne vrednosti so $\forall \lambda_j \geq 0$.

$Hf > 0$: pozitivno definitna $\quad \langle (Hf)v, v \rangle > 0 \ \forall v \neq 0 \iff$ — " — $\quad \forall \lambda_j > 0$.

$Hf \leq 0$: negativno semidefinitna

$Hf < 0$: negativno definitna

Kako je pri analizi 1: $f'(a) = 0$ $\quad$ lokalni maksimum $\Rightarrow f''(a) \leq 0$

$+$ $\quad$ lokalni minimum $\Rightarrow f''(a) \geq 0$

Kaj pa zadostni pogoji? $f''(a) < 0 \Rightarrow$ lokalni maksimum

$f''(a) > 0 \Rightarrow$ lokalni minimum

Trditev: Naj bo $D^{odp} \subseteq \mathbb{R}^n$, $f \in C^2(D)$ in $a \in D$.

i) Če je a lokalni maksimum za f, je $(Df)(a) = 0$ in $(Hf)(a) \leq 0$

ii) Če je a lokalni minimum za f je $(Df)(a) = 0$ in $(Hf)(a) \geq 0$.

i) podobno:

Dokaz: ii) $h \in \mathbb{R}^n$; $\|h\| = 1$.

$t \mapsto f(a+th)$ je definirana v okolici 0.

$\varphi(t) = f(a+th)$

Če ima f v a lokalni minimum, ima φ v 0 lokalni minimum. $\Rightarrow \varphi'(0) = 0$ in $\varphi''(0) \geq 0$.

$$\varphi'(0) = (Df)(a)\,h = \frac{\partial f}{\partial x_1}(a)h_1 + \dots + h_n \frac{\partial f}{\partial x_n}(a) = (D_h f)(a) = 0$$

$$\varphi'(t) = h_1 \frac{\partial f}{\partial x_1}(a+th) + \dots + h_n \frac{\partial f}{\partial x_n}(a+th)$$

$$\varphi''(t) = \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a+th)\,h_i h_j$$

$$\varphi''(0) = \langle (Hf)(a)h, h\rangle \geq 0 \quad (\text{Hessejna forma}) \quad = (D_h^2 f)(a)$$

$$\Rightarrow (Hf)(0) \geq 0 \quad \blacksquare$$

Izrek: Naj bo $f: D^{odp} \subseteq \mathbb{R}^n \to \mathbb{R}$ C^2 funkcija. Naj bo $a \in D$ stacionarna točka za f.

i) Če je $(Hf)(a) > 0$ ima f v a strogi lokalni minimum.

ii) Če je $(Hf)(a) < 0$ ima f v a strogi lokalni maksimum.

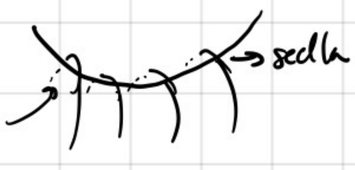
iii) Če ima $(Hf)(a)$ tako pozitivne kot tudi negativne lastne vrednosti, f v a nima lokalnega ekstrema.

Opomba: $n = 2$

ali $(x^2+y^2)\frac{1}{2}$ $\quad Hf = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0$

$(-x^2-y^2)\frac{1}{2}$ $\quad Hf = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$(x^2-y^2)\frac{1}{2}$ $\quad Hf = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Posledica: $n = 2$. Naj bo $f: D^{odp} \subseteq \mathbb{R}^n \to \mathbb{R}$ C^2 funkcija. Naj bo $(a,b) \in D$ stacionarna točka za f.

i) Če je $(f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$ ima f lokalni ekstrem v (a,b)

.) Če je $f_{xx}(a,b) > 0$ ima f v (a,b) lokalni minimum.

..) Če je $f_{xx}(a,b) < 0$ ima f v (a,b) lokalni maksimum.

ii) Če je $(f_{xx}f_{yy} - f_{xy}^2)(a,b) < 0$, f v (a,b) nima lokalnega ekstrema.

Dokaz: $(Hf)(a,b) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}(a,b)$ $\qquad \lambda, \mu$ lastni vrednosti je $\lambda\mu = \det(Hf)(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b)$

ii) Če je $\lambda\mu < 0$, sta nasprotno predznačeni $\Rightarrow f$ nima lokalnega ekstrema v (a,b).

i) Če je $\lambda\mu > 0$, ima λ, μ isti predznak. $\Rightarrow$ imamo lokalni ekstrem v (a,b).

iz $(f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0 \Rightarrow f_{xx}(a,b), f_{yy}(a,b)$ sta enako predznačeni

.) $f_{xx}(a,b) > 0 \Rightarrow f_{yy}(a,b) > 0 \Rightarrow \operatorname{tr}(Hf)(a,b) = (f_{xx} + f_{yy})(a,b) = \lambda + \mu > 0 \Rightarrow$ obe sta pozitivni

$\Rightarrow$ poz. definitnost $\Rightarrow$ lokalni minimum.

..) podobno (obratna smer).

Dokaz izreka: (iii) sledi iz trditve pred izrekom o potrebnih pogojih za lokalni ekstrem.

(ii) sledi iz (i) - simetrično (namesto f gledamo $-f$)

(i): $$f(a+h) = f(a) + \underbrace{\frac{1}{1!}(D_h f)(a)}_{0} + \underbrace{\frac{1}{2!}\langle (Hf)(a+\theta h)\,h, h\rangle}_{R_1}$$

$$f(a+h) - f(a) = \frac{1}{2!}\langle (Hf)(a+\theta h)h, h\rangle$$

$$\left(n=1: \quad \frac{1}{2} f''(a+\theta h)h^2 \right)$$

$$(Hf)(a+\theta h) = (Hf)(a) + E(h)$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a+\theta h) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a) + e_{ij}(h)$$

Ker je f C^2 velja: $\lim_{h \to 0} e_{ij}(h) = 0$ (zveznost drugih odvodov)

$$f(a+h)-f(a) = \underbrace{\frac{1}{2!}\langle (Hf)(a)h,h\rangle}_{\overset{\circ}{A}} + \frac{1}{2}\langle E(h),h\rangle \overset{?}{\geq} 0$$

$h := \frac{|h|\cdot h}{|h|}$; $h \neq 0$. $\qquad (>0)$

$= |h|\cdot v$; $v = \frac{h}{|h|}$ $|v|=1$

$$\Rightarrow \quad = \frac{|h|^2}{2}\left(\langle (Hf)(a)v,v\rangle + \langle E(h)v,v\rangle\right)$$

Funkcija: $v \mapsto \langle (Hf)(a)v,v\rangle > 0$ za vse $|v|=1$

je tudi na kompaktni sferi $\{v \mid |v|=1\}$. $\Rightarrow \exists M>0$, da je $\langle (Hf)(a)v,v\rangle \geq M > 0 \quad \forall v;\ |v|=1$

$\lim_{h\to 0}\langle E(h)v,v\rangle = 0$ enakomerno na $\{v;\ |v|=1\}$

$\Rightarrow \exists \delta>0 \ \forall |h|<\delta$ velja: $|\langle E(h)v,v\rangle| \leq \frac{M}{2} \quad \forall |v|=1.$

$$f(a+h)-f(a) \geq \frac{|h|^2}{2}\left(M-\frac{M}{2}\right) \text{ za vse } |h|<\delta.$$

$\geq \frac{1}{4}|h|^2 \cdot M$ je pozitivno. (strogo)

$\Rightarrow$ f ima v a strogi lokalni minimum

Zgled: $f(x,y,z) = x^2+y^2+z^2+2xyz$

f_x: $2x+2yz=0 \qquad x+yz=0 \Rightarrow x=-yz$

f_y: $2y+2xz=0 \Rightarrow y+xz=0 \qquad \Rightarrow y-yz^2=0 \Rightarrow y(1-z^2)=0$

f_z: $2z+2xy=0 \qquad z+xy=0 \qquad \Rightarrow z-y^2z=0 \Rightarrow z(1-y^2)=0$

i) $y=0, z=0, x=0$

ii) $y\neq 0$. $z^2=1$, $y^2=1 \Rightarrow (-1,1,1) \quad (-1,-1,-1)$

$(1,-1,1) \quad (1,1,-1)$

Imamo 5 stacionarnih točk.

Drugi odvodi:

$$(Hf)(x,y,z) = 2\begin{bmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{bmatrix}$$

$f(t,t,t) = 3t^2+2t^3$

ni globalnega min in max.

i) v $(0,0,0)$ je $(Hf)(0,0,0) = 2\cdot I > 0$, f ima v $(0,0,0)$ lokalni minimum.

ii) v $(-1,1,1)$ je $(Hf)(-1,1,1) = 2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2-1\right) - (1-\lambda+1) + (-1-1+\lambda)$$

$$= (1-\lambda)(\lambda^2-2\lambda) - (2-\lambda) + \lambda - 2$$

$$= (1-\lambda)(\lambda^2-2\lambda) - 2 + \lambda + \lambda - 2$$

$$= -\lambda^3 + 3\lambda^2 - \cancel{2\lambda} + \cancel{2\lambda} - 4$$

$$= -\lambda^3 + 3\lambda^2 - 4 = 0$$

$\left.\begin{array}{l}\lambda_1 = -1 \\ \lambda_2 = 2\end{array}\right\}$ Ta točka ni lokalni ekstrem!

Ostale točke obravnavamo podobno.

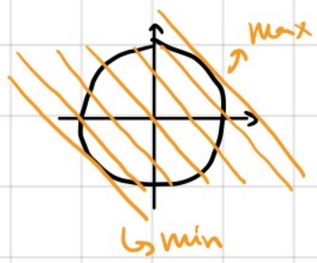
Vezani ekstremi: $M = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1,\ x+y+z=0\}$ → kompaktna množica

$f(x,y,z) = z$ (globalno nima ekstrema)

Ekstremi f na M?

Zgled: $M = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$

$f(x,y) = x+y$ Maksimum in minimum f na M?



Pogledamo nivojnice: v ekstremih sta nivojnica tangenti na M.

Izrek: Naj bo $D^{odp} \subseteq \mathbb{R}^n$, naj bodo $f, g_1, \dots, g_m \in C^1(D)$, $m < n$. Naj bo $G = (g_1, \dots, g_m): D \to \mathbb{R}^m$ maksimalnega ranga: $\operatorname{rang} G = m$. Naj bo $M = G^{-1}(\{0\}) \neq \emptyset$. Torej $M = \{x \in D \mid g_1(x) = 0, \dots, g_m(x) = 0\}$, je podmnogoterost v D. Naj ima $f: D \to \mathbb{R}$ v točki $a \in M$ lokalni ekstrem (kot funkcija $M \to \mathbb{R}$) Tedaj obstajajo take realne konstante, da je $(Df)(a) = \lambda_1 (Dg_1)(a) + \dots + \lambda_m (Dg_m)(a) = \sum_1^m \lambda_j (Dg_j)(a)$

so vezi - poi teh pogojih iščemo ploščino.

Opomba: <u>Lagrangeva metoda iskanja vezanih ekstremov:</u>

Tvorimo funkcijo $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(x) - \sum_1^m \lambda_j g_j(x)$ in iščemo stacionarne točke F.

$$\left.\begin{aligned} (D_x F) &= (Df) - \sum_1^m \lambda_j (Dg_j)(x) && (n \text{ enačb}) \\ (D_{\lambda_j} F) &= -g_j(x) = 0 \quad j = 1, \dots, m && (m \text{ enačb}) \end{aligned}\right\} \text{ za } n+m \text{ neznank}$$

$\lambda_1, \dots, \lambda_m$ so Lagrangeevi multiplikatorji

Zgled: $\left.\begin{aligned} M: \; & x^2 + y^2 + z^2 = 1 \\ & x + y + z = 0 \end{aligned}\right\}$ pogoja

$f(x, y, z) = z$.

$F(x, y, z) = z - \lambda(x^2 + y^2 + z^2 - 1) - \mu(x + y + z)$

$F_x: \; -\lambda(2x) - \mu = 0$

$F_y: \; -\lambda 2y - \mu = 0$

$F_z: \; 1 - 2\lambda z - \mu = 0$

$F_\lambda: \; x^2 + y^2 + z^2 - 1 = 0$

$F_\mu: \; -(x + y + z) = 0$

1.) $\lambda \neq 0$: $\Rightarrow x = -\frac{\mu}{2\lambda}$

$\Rightarrow y = -\frac{\mu}{2\lambda}$

$\Rightarrow z = \frac{1-\mu}{2\lambda}$

$x + y + z = \frac{1 - 3\mu}{2\lambda} = 0 \Rightarrow \mu = \frac{1}{3}$

$x^2 + y^2 + z^2 = 1 \iff 2 \cdot \frac{\mu^2}{4\lambda^2} + \frac{(1-\mu)^2}{4\lambda^2} = 1$

$\iff \frac{1}{9} + \frac{1}{9} + \frac{4}{9} = 4\lambda^2$

$\iff \frac{1}{6} = \lambda^2$

$\lambda_1 = \frac{1}{\sqrt{6}} \quad \lambda_2 = -\frac{1}{\sqrt{6}}$ (Dobimo dve točki)

Točki:

$$\left.\begin{aligned} x &= \frac{-\frac{1}{3}}{\frac{2}{\sqrt{6}}} = -\frac{\sqrt{6}}{6} \\ y &= \frac{-\frac{\sqrt{6}}{6}}{\frac{2}{3}} \\ z &= \frac{\frac{1}{3}}{2 \cdot \frac{1}{\sqrt{6}}} = \frac{\sqrt{6}}{3} \to \max \end{aligned}\right\} T_1 \qquad \left.\begin{aligned} x &= \frac{\sqrt{6}}{6} \\ y &= \frac{\sqrt{6}}{6} \\ z &= -\frac{\sqrt{6}}{3} \to \min \end{aligned}\right\} T_2$$

Dokaz izreka: Poglejmo preslikavo $F(x) = (f(x), g_1(x), \dots, g_m(x))$

$D \subseteq \mathbb{R}^n \to \mathbb{R}^{m+1} \qquad m+1 \leq n$

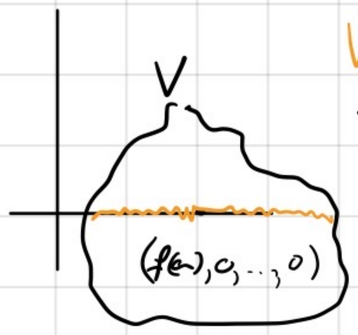
$F(a) = (f(a), 0, \dots, 0)$

Odvod:

$$(DF) = \begin{bmatrix} Df \\ Dg_1 \\ \vdots \\ Dg_m \end{bmatrix} \text{ gotovo neodvisni}$$

Želimo videti, da je $\operatorname{rang}(DF) = m$. Ker je vseh m vrstic lin. neodvisnih (po pogojih) $\Rightarrow (Df)(a)$ je lin. kombinacija $(Dg_1)(a), \dots, (Dg_m)(a)$. (To je naša trditev).

Denimo, da je $\operatorname{rang}(DF)(a) = m+1$, potem je F surjektivna iz okolice $a \in U \in \mathbb{R}^n$ v okolico $(f(a), 0, \dots, 0)$ v $\mathbb{R}^{m+1}$



Vse te točke so gline v M.

$\Rightarrow$ $\exists \varepsilon > 0$, da je množica $\{(y_1, 0, \dots, 0);\ |y_1 - f(a)| < \varepsilon\} \subset V$ v zalogi vrednosti preslikave F.

To pomeni, da f na M v okolici a zavzame večji in manjši vrednosti od $f(a)$. $\ast$

(Ne more biti lokalni ekstrem

$\blacksquare$

Zgled: $f(x,y) = x^2 - xy + y^2 - 3x + 4$

$x^2 + y^2 = 3$

$F(x,y,\lambda) = x^2 - xy + y^2 - 3x + 4 - \lambda(x^2 + y^2 - 3)$

F_x: $2x - y - 3 - 2\lambda x = 0 \Rightarrow 2x(1-\lambda) = y - 3$ $\qquad 2(1-\lambda) =: \alpha$

F_y: $-x + 2y - 2y\lambda = 0 \Rightarrow 2(1-\lambda)y = x$

F_λ: $x^2 + y^2 = 3$

$\alpha x = y - 3$

$y\alpha = x$ $\Rightarrow \alpha^2 y = y - 3 \Rightarrow 3 = y(1-\alpha^2) \Rightarrow y = \dfrac{3}{1-\alpha^2},\ x = \dfrac{3\alpha}{1-\alpha^2}$

$9\alpha^2 + 9 = 9(1-\alpha^2)^2$

$\alpha^2 + 1 = (\alpha^4 - 2\alpha^2 + 1)$

$3\alpha^2 = \alpha^4$: $\quad$ i) $\alpha = 0 \Rightarrow (0,3)$

ii) $\alpha = \pm\sqrt{3}$. $\Rightarrow \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$ in $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

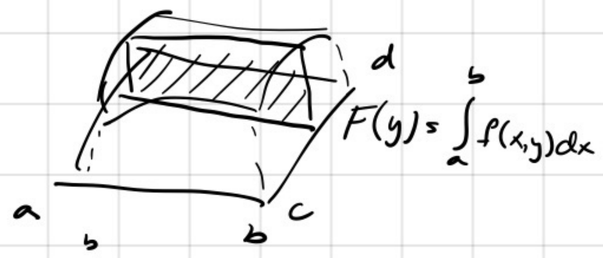
Integrali

Integral s parametrom: $f:[a,b]_x \times [c,d]_y \to \mathbb{R}$

$$F(y) = \int_a^b f(x,y)\,dx \qquad y\text{-parameter}$$

Lastnosti $f \rightsquigarrow$ Lastnosti F?

(bolj splošno: $F(y) = \int_{\alpha}^{\beta} f(x,y)\,dx$)



Opomba: $\int f(x,y)\,dx \rightsquigarrow \mathcal{F}_1(y) + \mathcal{F}_2(y) + \dots + \mathcal{F}_N(y)$

$\int_{\alpha}^{\infty} f(x,y)\,dx \rightsquigarrow \sum_1^{\infty} \mathcal{F}_i(y)$

Zgled: • $F(y) = \int_0^{\pi} \sin(xy)\,dx = \begin{cases} 0 & ; y=0 \\ -\frac{1}{y}\cos(xy)\Big|_0^{\pi} = \frac{1}{y}\left(1-\cos(\pi y)\right) & ; y \neq 0 \end{cases}$ (je gladka funkcija)

• Eulerjeva gama funkcija: $\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}\,dx = \int_0^1 x^{s-1}e^{-x}\,dx + \int_1^{\infty} x^{s-1}e^{-x}\,dx$

$D_\Gamma =$ i) $\int_1^{\infty} x^{s-1}e^{-x}\,dx \Rightarrow$ obstaja $\forall s \in \mathbb{R}$

$e^{-x}x^{s-1} = \underbrace{x^{s-1}e^{-\frac{x}{2}}}_{\text{omejen na } [1,\infty)\ \forall s} e^{-\frac{x}{2}} \leq M e^{-\frac{x}{2}}$

ii) $\int_0^1 x^{s-1}e^{-x}\,dx = \int_0^1 \frac{e^x}{x^{1-s}}\,dx$ obstaja $1-s<1 \Rightarrow \underline{\underline{s>0}}$

$\Rightarrow D_\Gamma = (0,\infty)$

$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}\,dx$

• Predznak: pozitivna funkcija: $\Gamma(s) > 0 \ \forall s \in D_\Gamma$

• $\Gamma(1) = 1$

• Osnovna rekurzivna relacija za Γ: $\Gamma(s+1) = \int_0^{\infty} \underbrace{x^s}_{u}\,\underbrace{e^{-x}dx}_{dv} = \overset{0}{\cancel{-x^s e^{-x}\Big|_0^{\infty}}} + s\int_0^{\infty} x^{s-1}e^{-x}\,dx = s\Gamma(s)$

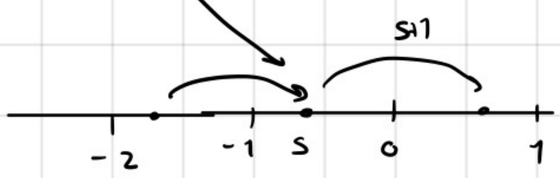
$du = sx^{s-1} \quad v = -e^{-x}$

$\Rightarrow \boxed{\Gamma(s+1) = s\Gamma(s)}$

$\Rightarrow \Gamma(n+1) = n\Gamma(n) = \dots = n!$

Opomba: $\Rightarrow \boxed{\Gamma(s) = \frac{\Gamma(s+1)}{s} \quad \forall s>0.}$

Ko $s \approx 0$ je $\Gamma(s) \approx \frac{1}{s}$



$\Gamma(-\frac{1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}} = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$

Iterativno lahko razširimo na $\mathbb{R} \setminus \{0 \cup \mathbb{Z}^-\}$

Definicija: Podmnožica $D \subseteq \mathbb{R}^n$ je lokalno kompaktna, če za vsako $a \in D$ obstaja tak $r>0$, da je $D \cap \overline{K(a,r)}$ kompaktna podmnožica

Zgledi: i) Vsaka zaprta množica je lokalno kompaktna

ii) Vsaka odprta množica je lokalno kompaktna

iii) $D \subseteq \mathbb{R}^2$ $D = K(0,1) \cup \{1,0\}$

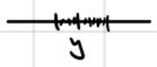
(: ni kompaktno

Trditev: Naj bo $D \subseteq \mathbb{R}^n$ lokalno kompaktna podmnožica. Naj bo I zaprt interval na $\mathbb{R}$. Naj bo $f: I_x \times D_y \to \mathbb{R}$ zvezna. Tedaj je

$$F(u,v,y) = \int_u^v f(x,y)\,dx \;;\; (u,v,y) \in I \times I \times D \quad \text{zvezna na } I \times I \times D.$$

Posledica trditve: D kot prej, $I=[a,b]$, f kot v trditvi. Potem je $F(y) = \int_a^b f(x,y)\,dx$ zvezna na D.

Dokaz trditve: Naj bo $(u_0, v_0, y_0) \in I \times I \times D$. Želimo pokazati zveznost F v tej točki



Vemo: D je lokalno kompaktna. Obstaja $r>0$, da je $D \cap \overline{K(y_0,r)}$ kompaktna množica.

$\Rightarrow I \times (D \cap \overline{K(y_0,r)})$ je kompaktna

•) f je na tej množici omejena: $\exists M$, da je $|f(x,y)| \le M \;\; \forall x,y \in I \times (D \cap \overline{K(y,r)})$

••) f je enakomerno zvezna na tej množici.

i) Naj bo $\varepsilon > 0$. Ker je f na $I_x \times (D \cap \overline{K(y,r)})_y$ enakomerno zvezna obstaja $\delta > 0$, da za $|y_1 - y_2| < \delta$ in $\forall x: ((x,y_1),(x,y_2)) \in I \times (D \cap \overline{K(y,r)})$ velja, da je $|f(x,y_1) - f(x,y_2)| < \frac{\varepsilon}{3(b-a)}$

ii) Naj velja tudi $|u-u_0| < \frac{\varepsilon}{3M}$ (*)

$|v-v_0| < \frac{\varepsilon}{3M}$

$$|F(u,v,y) - F(u_0,v_0,y_0)| = \left| \int_u^v f(x,y)\,dx - \int_{u_0}^{v_0} f(x,y_0)\,dx \right| = \left| \int_{v_0}^{v} f(x,y)\,dx + \int_u^{u_0} f(x,y)\,dx + \int_{u_0}^{v_0} f(x,y)\,dx - \int_{u_0}^{v_0} f(x,y_0)\,dx \right| \le$$

$$\left| \int_{v_0}^{v} f(x,y)\,dx \right| + \left| \int_u^{u_0} f(x,y)\,dx \right| + \left| \int_{u_0}^{v_0} |f(x,y) - f(x,y_0)|\,dx \right|$$

$$\le M|v-v_0| + M|u-u_0| + \frac{\varepsilon}{3(b-a)} \int_a^b 1\,dx \quad |y-y_0|<\delta$$

$$\overset{(*)}{\le} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \underline{\underline{\varepsilon}} \quad \checkmark$$

■

Odvajanje integralov s parametri

Trditev: Naj bo $f: [a,b]_x \times (c,d)_y \to \mathbb{R}$ zvezna. Naj bo za vsak $(x,y) \in [a,b] \times (c,d)$ f parcialno odvedljiva na y: $\frac{\partial f}{\partial y}(x,y)$. Naj bo tudi $\frac{\partial f}{\partial y}$ zvezna na pravokotniku. Potem je

$F(y) = \int_a^b f(x,y)\,dx$ odvedljiva funkcija na (c,d) in (zvezno odvedljiva, po prejšnji trditvi)

$$F'(y) = \frac{dF}{dy} = \frac{d}{dy} \int_a^b f(x,y)\,dx = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx$$

Posledica: Naj bo f kot prej, naj bosta $\alpha, \beta: (c,d) \to [a,b]$ zvezno odvedljivi.

Tedaj:
$$\frac{d}{dy}\left(\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx \right) = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x,y)\,dx + \beta'(y) f(\beta(y),y) - \alpha'(y) f(\alpha(y),y)$$

Dokaz posledice: $F(u,v,y) = \int_u^v f(x,y)\,dx$

$$\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx = F(\alpha(y), \beta(y), y)$$

Odvajamo: $F_u \alpha' + F_v \beta' + F_y$ $\quad (\alpha(y), \beta(y), y)$

$F_v = f(v,y)$

$F_u = -f(u,y)$

$$F_y = \int_u^v \frac{\partial f}{\partial y}(x,y)\,dx$$

Posledica: Naj bo $D^{odp} \subseteq \mathbb{R}^n$. Naj bo $f: [a,b] \times D \to \mathbb{R}$ zvezna.

Denimo, da za vsako točko $(x,y) \in [a,b] \times D$ in vsak $j = 1, \dots, n$ obstaja parcialni odvod $\frac{\partial f}{\partial y_j}$ in so vsi zvezne funkcije na $[a,b] \times D$.

Potem je $F(y) = \int_a^b f(x,y)\,dx$, F je $C^1(D)$ in velja,

$$\frac{\partial F}{\partial y_j} = \int_a^b \frac{\partial f}{\partial y_j}(x,y)\,dx$$

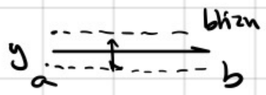
$\frac{\partial f}{\partial y}(x,y^*)\cdot|h|$; y^* med y in $y+h$

Dokaz trditve: $y \in (c,d)$ $\quad \frac{F(y+h)-F(y)}{h} = \frac{1}{h}\int_a^b (f(x,y+h)-f(x,y))\,dx =$

$$= \int_a^b \frac{\partial f}{\partial y}(x,y^*)\,dx$$

$[y-r \quad y+r] \subseteq (c,d)$

Oglejmo si razliko $\left|\frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx\right| = \left|\int_a^b \left(\frac{\partial f}{\partial y}(x,y^*) - \frac{\partial f}{\partial y}(x,y)\right)dx\right| \le \int_a^b \left|\frac{\partial f}{\partial y}(x,y^*) - \frac{\partial f}{\partial y}(x,y)\right|dx$



Naj bo $r>0$, da bo $[y-r, y+r] \subset (c,d)$. Množica $[a,b]\times[y-r,y+r]$ je kompaktna. Vemo: $\frac{\partial f}{\partial y}$ je zvezna na $[a,b]\times(c,d)$, Torej je enakomerno zvezna na $[a,b]\times[y-r,y+r]$.

Naj bo $\varepsilon>0$. $\exists\, \delta>0$, da za $|h|<\delta \Rightarrow \left|\frac{\partial f}{\partial y}(x,y_1) - \frac{\partial f}{\partial y}(x,y_2)\right| < \frac{\varepsilon}{b-a}$ za vse $(x,y_1),(x,y_2) \in [a,b]\times[y-r,y+r]$, kjer je $|y_1-y_2| = |h| < \delta$.

Torej za $|h|<\delta \Rightarrow |y^*-y|<\delta \Rightarrow \left|\frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx\right| \le \left|\int_a^b \frac{\partial f}{\partial y}(x,y^*) - \frac{\partial f}{\partial y}(x,y)\,dx\right| < \frac{\varepsilon}{b-a}(b-a) = \varepsilon$ ■

Zgled: $F(a) = \int_0^1 \frac{dx}{x^2+a^2}$, $a>0$

$$F(a) = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right)\Big|_0^1 = \frac{1}{a}\operatorname{arctg}\left(\frac{1}{a}\right)$$

Odvod:

$$\cdot)\ F'(a) = \int_0^1 \frac{\partial}{\partial a}\frac{1}{x^2+a^2}\,dx = \int_0^1 \frac{-2a}{(x^2+a^2)^2}\,dx$$

ali

$$\cdot\cdot)\ F'(a) = \frac{\frac{1}{1+\frac{1}{a^2}}\left(-\frac{1}{a^2}\right)\cdot a - \operatorname{arctg}\left(\frac{1}{a}\right)}{a^2}$$

$$\Rightarrow \int_0^1 \frac{dx}{(x^2+a^2)^2} = \frac{\frac{a}{a^2+1} + \operatorname{arctg}\left(\frac{1}{a}\right)}{2a^3}$$

Za $a=1$: $\int_0^1 \frac{1}{(x^2+1)^2}\,dx = \frac{\frac{1}{2}+\frac{\pi}{4}}{2} = \underline{\underline{\frac{2+\pi}{8}}}$

$$G(a) = \int_0^a \frac{dx}{x^2+a^2}$$

$$G(a) = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right)\Big|_0^a = \frac{\pi}{4a}$$

$$\cdot)\ G'(a) = -\frac{\pi}{4a^2}$$

$$\cdot\cdot)\ G'(a) = -2a\int_0^a \frac{dx}{(x^2+a^2)^2} + 1\,\frac{1}{2a^2} = -\frac{\pi}{4a^2}$$

$$\int_0^a \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{8a^3} + \frac{1}{4a^3} = \frac{\pi+2}{8a^3}$$

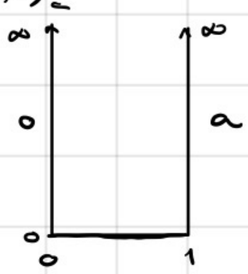
$a=1$: $G(1) = \frac{\pi+2}{8}$ ✓

Zgled: $\int_0^1 \frac{x-1}{\ln x}\,dx$ $\quad F(a) = \int_0^1 \frac{x^a-1}{\ln x}\,dx$ $\quad a \ge 0$

$$\lim_{x\to 1} \frac{x^a-1}{\ln x} = \lim_{x\to 1} \frac{a x^{a-1}}{\frac{1}{x}} = \lim_{x\to 1} a x^a = a$$

$f(x,a) = \frac{x^a-1}{\ln x}$ $\quad [0,1]_x \times [0,\infty)_a$

$f(x,a)$ je zvezna funkcija:



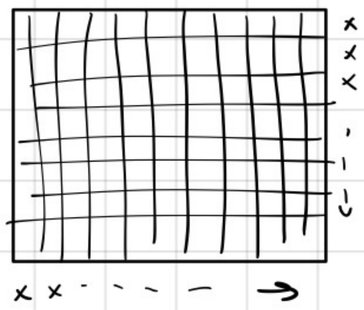
$$\frac{\partial f}{\partial a} = \frac{1}{\ln x}\left(x^a \ln x\right) = x^a$$

F je odvedljiva in $F'(a) = \int_0^1 x^a\,dx = \frac{1}{a+1}x^{a+1}\Big|_0^1 = \underline{\underline{\frac{1}{a+1}}} \Rightarrow F(a) = \ln(a+1) + C$

Za $a=0$ je $F(0) = 0 = \ln 1 + C \Rightarrow \underline{\underline{C=0}}$.

$\Rightarrow F(a) = \ln(a+1) \Rightarrow F(1) = \ln 2.$

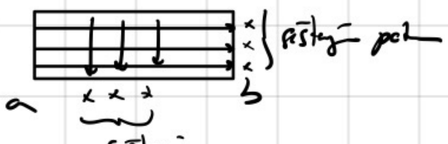
Integral integrala s parametrom:



$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$

Dvakratni integrali, ne dvojni integral

Izrek: Naj bo $f: [a,b]\times[c,d] \to \mathbb{R}$ zvezna. Tedaj je $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$



(Fubinijev izrek)

Dokaz: $\Psi(y) = \int_c^y \left(\int_a^b f(x,s)\,dx \right) ds \qquad \Phi(y) = \int_a^b \left(\int_c^y f(x,s)\,ds \right) dx \qquad y \in [c,d].$

Radi bi: $\Psi(d) = \Phi(d)$. (tudi $\forall y \in [c,d]$).

$\left.\begin{array}{l} \cdot) \text{ Velja: } \Psi(c) = \Phi(c) = 0. \\ \cdot\cdot) \text{ Če so odvodi enaki: } \Psi' = \Phi' \end{array}\right\} \Rightarrow \Psi = \Phi$

Enaka odvoda:

i) $\Psi(y) = \int_c^y \left(\int_a^b f(x,s)\,dx \right) ds$

zvezna na $[c,d]$ $\quad F(s) = \int_a^b f(x,s)\,dx \Rightarrow \Psi(y) = \int_c^y F(s)\,ds \Rightarrow \Psi$ je odvedljiva in $\Psi'(y) = \int_a^b f(x,y)\,dx = F'(y)$

ii) $\Phi(y) = \int_a^b \underbrace{\left(\int_c^y f(x,s)\,ds \right)}_{} dx = \int_a^b g(x,y)\,dx$

$g(x,y) = \int_c^y f(x,s)\,ds$, g je zvezna funkcija obeh argumentov na $[a,b]\times[c,d]$.

Ali $\frac{\partial g}{\partial y}$ obstaja? $\frac{\partial g}{\partial y}(x,y) = f(x,y)$

$\Rightarrow \Phi'$ obstaja in $= \int_a^b \frac{\partial g}{\partial y}(x,y)\,dx = \int_a^b f(x,y)\,dx$

Ker sta odvoda enaka $\Rightarrow \Psi = \Phi$.

Zgled: $\int_0^1 \left(\int_0^2 (x+y^2)\,dx \right) dy = \int_0^1 \frac{1}{2}x^2 + y^2 x \Big|_0^2 dy = \int_0^1 2 + 2y^2\,dy = 2y + \frac{2}{3}y^3 \Big|_0^1 = \underline{\underline{\frac{8}{3}}}$

$\int_0^2 \left(\int_0^1 (x+y^2)\,dy \right) dx = \int_0^2 \left(xy + \frac{1}{3}y^3 \right)\Big|_0^1 dx = \int_0^2 x + \frac{1}{3}\,dx = \frac{x^2}{2} + \frac{1}{3}x \Big|_0^2 = 2 + \frac{2}{3} = \underline{\underline{\frac{8}{3}}}.$

Posplošeni integrali s parametrom:

Y množica, $a \in \mathbb{R}$ in $f: [a,\infty)_x \times Y_y \to \mathbb{R}$

Naj bo f $\forall y \in Y$ zvezna funkcija na $[a,\infty)$. Naj $\forall y \in Y$ obstaja $F(y) = \int_a^\infty f(x,y)\,dx$

Opomba: $\int_{-\infty}^b$ pače tudi v to. $\int_{-\infty}^\infty f(x,y)\,dx$ razbijemo na dva kosa $\left(\int_{-\infty}^0 + \int_0^\infty \right)$

$F(y) = \int_a^\infty f(x,y)\,dx = \lim_{b\to\infty} \int_a^b f(x,y)\,dx$

$\lim_{b\to\infty} \int_b^\infty f(x,y)\,dx = 0.$

$\sum_1^\infty f_j(y)$ enakomerna konvergenca

Definicija: Integral $F(y) = \int_a^\infty f(x,y)\,dx$ konvergira enakomerno na Y, če $\forall \varepsilon > 0\ \exists b_0 \geq a$, da za vsak $b \geq b_0$ in vsak $y \in Y$ velja: $\left| \int_b^\infty f(x,y)\,dx \right| < \varepsilon$

Zgled: $F(y) = \int_0^\infty y e^{-xy}\,dx$, $Y = [0,\infty)$ $\qquad \int_0^\infty y e^{-xy}\,dx = -e^{-xy}\Big|_0^\infty = 1$

$F(y) = \begin{cases} 0 & ; y = 0 \\ 1 & y > 0 \end{cases}$ (ni zvezna funkcija)

i) $c>0$, $y\in[c,\infty)$, smo stran od nicle. $b\geq b_0$

$$\int_b^\infty ye^{-xy}dx = -e^{-xy}\Big|_b^\infty = e^{-by} \leq e^{-bc} \leq e^{-b_0\cdot c} < \varepsilon$$

Za dani ε obstaja tak b_0, da je $e^{-b_0c}<\varepsilon$.

$\Rightarrow \left|\int_b^\infty ye^{-xy}dx\right| < \varepsilon \quad \forall b\geq b_0,\ \forall y\geq c.$

$F(y)=\int_0^\infty ye^{-xy}dx$ enakomerno konvergira na $[c,\infty)$ $\forall c>0$.

ne konvergira enakomerno na $(0,\infty)$

konvergira na $[0,\infty)$.

ii) $y\in(0,\infty)$

$$\int_b^\infty ye^{-xy}dx = -e^{-xy}\Big|_b^\infty = e^{-by}$$

$y=\frac{1}{b}>0$, ne bo poljubno majhno.

Vedno lahko najdemo $y_1\ (\frac{1}{b})$ ne glede na b, da bo $\left|\int_b^\infty ye^{-xy}dx\right| > \frac{1}{2e}$

$\Rightarrow$ Konvergenca ni enakomerna na $(0,\infty)$.

$F(y)=\int_a^\infty f(x,y)dx$ Velja $\lim_{b\to\infty} F_b(y) = F(y)$ enakomerno na Y.

$F_b(y)=\int_a^b f(x,y)dx$

Trditev: Naj bo $Y\subseteq\mathbb{R}^n$ lokalno kompaktna, naj bo $f:[a,\infty)\times Y\to\mathbb{R}$ zvezna in naj integral s parametri $F(y)=\int_a^\infty f(x,y)dx$ konvergira enakomerno na Y. Potem je F zvezna na Y.

Dokaz: Vemo: $F_b(y)=\int_a^b f(x,y)dx$ so zvezne na Y.

Poleg tega je $\lim_{b\to\infty}F_b(y)=F(y)$ enakomerno na Y (integral enakom. konv.)

Torej je F zvezna na Y. (enakomerna limita zveznih funkcij) ■

Opomba: zveznost je lokalna lastnost

F je zv. na Y či je zv. v vsaki točki y oz.

či je zv. v okolici vsake točke $y\in Y$.

Po prejšnji trditvi je za zv. F na Y dovolj zahtevati, da je integral lokalno enakomerno konvergentan.

$\forall y\in Y\ \exists r>0$, da je integral enakomerno konvergentan na $Y\cap K(y,r)$

($\Gamma(s)$ je zvezna na $(0,\infty)$.)

($\sum_1^\infty f_j(y)$ $\quad |f_j(y)|\leq a_j$ in $\sum a_j<\infty$, je $\sum f_j(y)$ enakom. konv.)

Trditev: Naj bo $f:[a,\infty)\times Y\to\mathbb{R}$ zvezna $\forall y\in Y$. Denimo, da obstaja taka zvezna funkcija $g:[a,\infty)\to(0,\infty)$, da $\forall(x,y)\in[a,\infty)\times Y$ velja: $|f(x,y)|\leq g(x)$ in da obstaja integral $\int_a^\infty g(x)dx$. Potem $F(y)=\int_a^\infty f(x,y)dx$ konvergira enakomerno na y.

Dokaz: $\left|\int_b^\infty f(x,y)dx\right| \leq \int_b^\infty |f(x,y)|dx \leq \int_b^\infty g(x)dx$

Naj bo $\varepsilon>0$. Ker $\int_a^\infty g(x)dx$ obstaja, obstaja tak b_0, da $\forall b\geq b_0$ velja: $\left|\int_b^\infty g(x)dx\right|<\varepsilon$

$\Rightarrow \left|\int_b^\infty f(x,y)dx\right|<\varepsilon \quad \forall y\in Y.$ ■

Zgled: $\int_1^\infty x^{s-1}e^{-x}dx$ → zv. funkcija na $\mathbb{R}$ $\quad M>0$, $s\in[-M,M]$. Potem je $x^{s-1}e^{-x}\leq x^{M-1}e^{x} =: g(x)$.

g je integrabilna na $[1,\infty)$

$\Rightarrow$ integr. $s\mapsto \int_1^\infty x^{s-1}e^{-x}dx$ konvergira enakomerno na $[-M,M]$

$\Rightarrow \int_1^\infty x^{s-1}e^{-x}dx$ konvergira lokalno enakomerno na $\mathbb{R}$.

Kaj pa $\int_0^1 x^{s-1}e^{-x}dx$? $\quad x=t^N$

($[c,M]$ $\quad 0<c<M$ $\quad x^{s-1}e^{-x}\leq x^{c-1}e^{-x}$ za $0<x\leq 1$ je integrabilna)

$$\int_0^1 x^{s-1}e^{-x}dx = \int_0^1 t^{N(s-1)}e^{-t^N}N\cdot t^{N-1}dt = \int_0^1 N\cdot t^{Ns-1}e^{-t^N}$$

$0<c<M$, $s\in[c,M]$

Za velik N bo $Ns-1>1$, ni singularnosti $\forall s\in[c,M]$.

Trditev: Naj bo $f:[a,\infty)\times[c,d]\to\mathbb{R}$ zvezna, naj integral $F(y)=\int_a^\infty f(x,y)\,dx$ konvergira enakomerno na $[c,d]$. Tedaj
$$\int_c^d\left(\int_a^\infty f(x,y)\,dx\right)dy=\int_a^\infty\left(\int_c^d f(x,y)\,dy\right)dx$$

Dokaz: $F(y)$ je zvezna funkcija, ker enakomerno konvergira. Zato obstaja $\int_c^d F(y)\,dy = \int_c^d \lim_{b\to\infty} F_b(y)\,dy \overset{\text{enakomerna}}{=} \lim_{b\to\infty}\int_c^d F_b(y)\,dy = \lim_{b\to\infty}\int_c^d\int_a^b f(x,y)\,dx\,dy$
$=\lim_{b\to\infty}\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$: ta limita obstaja, po def. $\Rightarrow \int_a^\infty\int_c^d f(x,y)\,dy\,dx$ ■

$$\int_c^\infty\left(\int_a^\infty f(x,y)\,dx\right)dy \overset{?}{=} \int_a^\infty\int_c^\infty f(x,y)\,dy\,dx \quad\leftrightsquigarrow\quad \sum_i\sum_j a_{ij}=\sum_j\sum_i a_{ij}$$ (rabimo absolutno konvergenco)

Trditev: Naj bo $f:[a,\infty)\times[c,\infty)\to[0,\infty)$ nenegativna zvezna funkcija. Denimo, da $F(y)=\int_a^\infty f(x,y)\,dx$ konvergira lokalno enakomerno na $[c,\infty)$ in da $G(x)=\int_c^\infty f(x,y)\,dy$ konvergira lokalno enakomerno na $[a,\infty)$. (F je zv. na $[c,\infty)$, G je zv. na $[a,\infty)$). F,G sta zvezni

Tedaj:
$$\int_c^\infty\left(\int_a^\infty f(x,y)\,dx\right)dy=\int_a^\infty\left(\int_c^\infty f(x,y)\,dy\right)dx.$$

Torej: ali sta oba enaka ∞, ali pa sta oba končna in enaka

Dokaz:
$$\int_a^b G(x)\,dx=\int_a^b\left(\int_c^\infty f(x,y)\,dy\right)dx \overset{\text{poljubni f.d.}}{=} \int_c^\infty\left(\int_a^b f(x,y)\,dx\right)dy\le\int_c^\infty\int_a^\infty f(x,y)\,dx\,dy=\int_c^\infty F(y)\,dy \quad(\forall b)$$
$$\Rightarrow \int_a^\infty G(x)\,dx\le\int_c^\infty F(y)\,dy.$$

Če je $\int_c^\infty F(y)\,dy$ končen $\Rightarrow \int_a^\infty G(x)\,dx$ je končen. Ponovimo zadnji stolpec, kjer vlogi F in G zamenjamo.

Podobno če je neskončen. ■

Velja tudi:

Trditev: Naj bo $f:[a,\infty)\times[c,\infty)\to\mathbb{R}$ zv. Naj bosta $\int_a^\infty|f(x,y)|\,dx$ lokalno enakomerno konvergentna na $[c,\infty)$ in $\int_c^\infty|f(x,y)|\,dy$ lokalno enakomerno konvergentna na $[a,\infty)$. Če je ali $\int_c^\infty\left(\int_a^\infty|f(x,y)|\,dx\right)dy$ končen ali $\int_a^\infty\left(\int_c^\infty|f(x,y)|\,dy\right)dx$ končen, je
$$\int_a^\infty\left(\int_c^\infty f(x,y)\,dy\right)dx=\int_c^\infty\left(\int_a^\infty f(x,y)\,dx\right)dy$$

(Brez dokaza - zapiski ga imajo)

$$\left(\sum_1^\infty f_n(y)\right)'=\sum_1^\infty f_n'(y)$$ konvergira f in enak. c. odvoda

Trditev (odvod): Naj bo $f:[a,\infty)\times(c,d)\to\mathbb{R}$. Naj $\int_a^\infty f(x,y)\,dx=F(y)$ konvergira na (c,d). Naj za vsak $(x,y)\in[a,\infty)\times(c,d)$ obstaja $\frac{\partial f}{\partial y}$, ki je tudi zvezna na $[a,\infty)\times(c,d)$. Naj $y\mapsto\int_a^\infty\frac{\partial f}{\partial y}(x,y)\,dx$ konvergira lokalno enakomerno na (c,d).
Tedaj je F zvezno odvedljiva na (c,d) in $F'(y)=\int_a^\infty\frac{\partial f}{\partial y}(x,y)\,dx$.

Dokaz: $G(y)=\int_a^\infty\frac{\partial f}{\partial y}(x,y)\,dx$ je zvezna na $[c,d]$.

$y_0\in(c,d)$
$$\left(\int_{y_0}^y G(s)\,ds\right)'\overset{\text{OIA}}{=}G(y)$$
$$\int_{y_0}^y G(s)\,ds=\int_{y_0}^y\left(\int_a^\infty\frac{\partial f}{\partial y}(x,s)\,dx\right)ds\overset{\text{lok. enak. konv.}}{=}\int_a^\infty\int_{y_0}^y\frac{\partial f}{\partial y}(x,s)\,dx\,ds=\int_a^\infty f(x,y)-f(x,y_0)\,dx=F(y)-F(y_0)$$
$$\int_{y_0}^y G(s)\,ds=F(y)-F(y_0)\Rightarrow F \text{ je odvedljiva in } F'=G.$$ ■

Zgled: $0<c<d$: $\int_c^d\left(\int_0^\infty e^{-xy}\,dx\right)dy=\int_c^d-\frac{1}{y}e^{-xy}\Big|_0^\infty dy=\int_c^d\frac{1}{y}\,dy=\ln d-\ln c=\ln\frac{d}{c}$
$$=\int_0^\infty\left(\int_c^d e^{-xy}\,dy\right)dx=\int_0^\infty-\frac{1}{x}e^{-xy}\Big|_c^d\,dx=\int_0^\infty\frac{e^{-cx}-e^{-dx}}{x}\,dx$$

Trditev: Naj bo $D^{\text{odp}}\subseteq\mathbb{R}^n$ in $f:[a,\infty)\times D\to\mathbb{R}$ zv. in naj za vsak $(x,y)\in[a,\infty)\times D$ obstaja $\frac{\partial f}{\partial y_j}\in C([a,\infty)\times D)$.
Naj za vsak $y\in D$ obstaja $\int_a^\infty f(x,y)\,dx=F(y)$. Naj za vsak $j=1,\dots,n$ $\int_a^\infty\frac{\partial f}{\partial y_j}(x,y)\,dx$ konvergira lokalno enakomerno na D.
Potem je $F\in C^1(D)$ in $\frac{\partial F}{\partial y_j}(y)=\int_a^\infty\frac{\partial f}{\partial y_j}(x,y)\,dx$ $\forall j$.

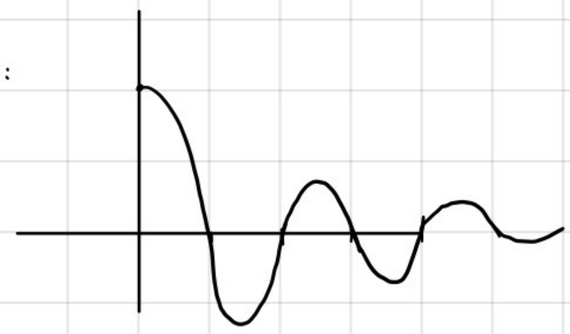
Zgled: $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Opombe: -) $\int_0^\infty \frac{\sin ax}{x}\,dx \overset{a>0}{\underset{ax=t}{=}} \int_0^\infty \frac{\sin(t)}{\frac{t}{a}}\frac{dt}{a} = \int_0^\infty \frac{\sin(t)}{t}\,dt = \frac{\pi}{2}$

$\overset{a<0}{=} -\int_0^\infty \frac{\sin(t)}{t}\,dt = -\frac{\pi}{2}$

$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\sin(ax)}{x}\,dx = \operatorname{sgn}(a)$

-) $\frac{\sin x}{x}$:



$\int_0^\infty \frac{|\sin x|}{x}\,dx = \infty$, ne obstaja

-) f zvezna na $[a,b]$, $g \in C^1([a,b])$ $g:[a,b]\to[0,\infty)$, g padajoča

$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\beta f(x)\,dx$ $\beta \in [a,b]$

Dokaz formule: $F(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$, $a \geq 0$. (Iščemo $F(0)$)

Za $0<c\leq a$ je integral enakomerno konvergentan. $\left(\left|e^{-ax}\frac{\sin x}{x}\right| \leq e^{-cx}\right) \Rightarrow F$ zvezna na $(0,\infty)$.

$F(0) = \lim_{a\downarrow 0} F(a) \Rightarrow$ vdj: $F \in C([0,\infty))$.

Integral $F(a)$ je enakomerno konvergentan na $[0,\infty)$.

$f(x) = \sin x$

$g(x) = e^{-ax}\cdot\frac{1}{x}$

ostanki $\int_\alpha^\beta e^{-ax}\cdot\frac{\sin x}{x}\,dx = \frac{e^{-a\alpha}}{\alpha}\cdot\int_\alpha^\gamma \sin(x)\,dx$, γ med α in β $\leq 2\cdot\frac{e^{-\alpha a}}{\alpha} \leq \frac{2}{\alpha}$ (je enakomerno konvergentna)

$\Rightarrow \left|\int_\alpha^\infty e^{-ax}\cdot\frac{\sin x}{x}\,dx\right| \leq \frac{2}{\alpha}$

Integral konvergira enakomerno na $[0,\infty)$, $F \in C([0,\infty))$.

$F(a) = \int_0^\infty \frac{e^{-ax}}{x}\sin x\,dx$

$F'(a) = \int_0^\infty -e^{-ax}\sin x\,dx$

Za $0<c\leq a$: $|e^{-ax}\sin x| \leq e^{-cx}$ (konvergira lokalno enakomerno na $(0,\infty)$)

F je odvedljiva na $(0,\infty)$ in $F'(a) = -\int_0^\infty e^{-ax}\sin x\,dx$

$\int e^{-ax}\sin x\,dx = \operatorname{Im}\left(\int e^{-ax}e^{ix}\,dx\right) = \operatorname{Im}\left(\int e^{(-a+i)x}\,dx\right) = \operatorname{Im}\left(\frac{1}{i-a}e^{(-a+i)x}\right)$

$F'(a) = \int_0^\infty -e^{-xa}\sin x\,dx = -\operatorname{Im}\left(\frac{-a-i}{a^2+1}\cdot e^{(-a+i)x}\Big|_0^\infty\right) = \operatorname{Im}\left(\frac{-a-i}{a^2+1}\right) = -\frac{1}{a^2+1}$ $F'(a) = -\frac{1}{a^2+1}$

$F(a) = -\operatorname{arctg} a + C$ → rabimo.

Vemo: $\left|e^{-ax}\frac{\sin x}{x}\right| \leq e^{-ax} \Rightarrow |F(a)| \leq \int_0^\infty e^{-ax}\,dx = \frac{1}{a}$

$\Rightarrow \lim_{a\to\infty} F(a) = 0$

$\lim_{a\to\infty}(C - \operatorname{arctg} a) = C - \frac{\pi}{2} = 0 \Rightarrow \underline{\underline{C = \frac{\pi}{2}}}$

$\Rightarrow F(a) = \frac{\pi}{2} - \operatorname{arctg} a$

$\Rightarrow F(0) = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Eulerjeva Γ in β funkcija:

$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$

- $D_\Gamma = (0,\infty)$
- $\Gamma(s+1) = s\Gamma(s)$
- $\Gamma(n+1) = n!$, $n\in\mathbb{N}$
- $\Gamma(1) = 1$
- Γ je zv. na $(0,\infty)$
- $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ $s>0$, $s\approx 0$ je $\Gamma(s)\approx\frac{1}{s}$.
- $\Gamma(s) \in C^\infty(0,\infty)$
- $\Gamma(s) > 0$
- $\Gamma''(s) = \int_0^\infty x^{s-1}\ln^2 x\, e^{-x}\,dx > 0 \Rightarrow \Gamma$ je konveksna funkcija.

(velja celo več: $\ln\Gamma(s)$ je konveksna.

Odvodi: $\Gamma'(s) = \int_0^\infty x^{s-1} \cdot \ln x \cdot e^{-x}\,dx = \int_0^1 x^{s-1} \ln x \cdot e^{-x}\,dx + \int_1^\infty x^{s-1} \ln x\, e^{-x}\,dx$

Za $0<c\le s\le d$: (lok. enak. konv.): Oba integrala konvergirata lokalno enakomerno na $(0,\infty)$.

$\Gamma^{(k)}(s) = \int_0^\infty x^{s-1} (\ln x)^k e^{-x}\,dx$ - tudi konvergira lokalno enakomerno. $\Rightarrow \Gamma \in C^\infty(0,\infty)$

Konveksna:

$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \quad \forall \lambda \in [0,1]$

$\lambda = \frac{1}{2}$: $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \quad \forall x,y$

iz tega sledi zgornjo za $\forall \lambda \in [0,1] \cap \mathbb{Q}$

+ zv. $\Rightarrow f$ je konveksna

$$\Gamma\left(\frac{s_1+s_2}{2}\right) = \int_0^\infty x^{\frac{s_1+s_2}{2}-1} e^{-x}\,dx = \int_0^\infty x^{\frac{s_1-1}{2}} e^{-\frac{x}{2}} x^{\frac{s_2-1}{2}} e^{-\frac{x}{2}}\,dx \overset{\text{Cauchy Schwarz}}{\le} \sqrt{\int_0^\infty x^{s_1-1} e^{-x}\,dx \cdot \int_0^\infty x^{s_2-1} e^{-x}\,dx}$$

$$\Rightarrow \Gamma\left(\frac{s_1+s_2}{2}\right) \le \sqrt{\Gamma(s_1)\Gamma(s_2)} \Rightarrow (\ln\Gamma)\left(\frac{s_1+s_2}{2}\right) \le \frac{\ln\Gamma(s_1) + \ln\Gamma(s_2)}{2}$$

$\Rightarrow \ln\Gamma$ je konveksna na $(0,\infty)$

(f, $\ln f$ je konveksna $\Rightarrow f$ konveksna)

($\ln f = h$, $f = e^h$: $f\left(\frac{x+y}{2}\right) = e^{h\left(\frac{x+y}{2}\right)} \le e^{\frac{1}{2}(h(x)+h(y))} = \sqrt{f(x)\cdot f(y)} \le \frac{1}{2}(f(x)+f(y))$) $\quad \sqrt{ab} \le \frac{1}{2}(a+b)$

Izrek Γ je natanko določena z lastnostmi:

i) $\Gamma(1) = 1$

ii) $\Gamma(s+1) = s\Gamma(s)$

iii) $\Gamma > 0$ in zvezna na $(0,\infty)$

iv) $\ln\Gamma$ je konveksna.

Ali je (iv) res nujna lastnost: $f(s) = \Gamma(s)\, e^{\sin(2\pi s)}$, logaritem ni konveksen, ne izpolnjuje (iv).

Velja tudi: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Zato lahko dobimo: $\Gamma\left(\frac{n}{2}\right)$

Npr. $\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$.

Zgled: $\int_0^\infty e^{-ax^2}\,dx$, $a>0$

($\int_0^\infty x^\alpha e^{-ax^\beta}\,dx$ se prevede na Γ)

$ax^2 = t \quad dx = \frac{\frac{1}{2} t^{1/2-1}}{\sqrt{a}}\,dt$

$$= \int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{a}} \cdot t^{1/2-1}\,dt = \frac{1}{2\sqrt{a}}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{4a}}$$

Zgled: $\int_{-\infty}^\infty e^{-\frac{(x-a)^2}{2\sigma^2}}\,dx$, $\sigma>0$, $a\in\mathbb{R}$: Normalna porazdelitev z avg. a in st. odklon $= \sigma$.

a je samo translacija $\quad = \int_{-\infty}^\infty e^{-\frac{x^2}{2\sigma^2}}\,dx = 2\cdot\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{2\sqrt{\pi}}{2\sqrt{\frac{1}{2\sigma^2}}} = \sqrt{2\pi\sigma^2} = \sigma\sqrt{2\pi}$

Normalna porazdelitev ima $\frac{1}{\sqrt{2\pi}\,\sigma}$ gostoto verjetnosti (da bo integral 1).

Beta: $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

$D_B = (0,\infty) \times (0,\infty)$

$B(p,q) = B(q,p) \quad (1-x = t)$

Za $x = \sin^2 t$, $1-x = \cos^2 t$ $\quad dx = 2\sin t\cos t\,dt$

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-2} t\,(\cos^{2q-2} t)\sin t\cos t\,dt$$

$$\int_0^{\frac{\pi}{2}} \sin^\alpha t \cos^\beta t\,dt = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right), \quad \alpha>-1, \beta>-1$$

$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Izrek: $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Zgled: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 t \cos^{10} t \, dt = \frac{1}{2} B\left(\frac{9}{2}, \frac{11}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{11}{2}\right)}{\Gamma(10)} = \frac{\frac{1}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{9!}$

Trditev: $\displaystyle B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt$

Dokaz trditve: $t = \dfrac{x}{1-x}$, $t - tx = x \Leftrightarrow x = \dfrac{t}{1+t}$ $\quad dx = \dfrac{1}{(1+t)^2} dt$

$$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p-1}} \frac{1}{(1+t)^{q-1}} \cdot \frac{1}{(1+t)^2} dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt \quad \blacksquare$$

Opomba: $B(p,1-p) = \dfrac{\pi}{\sin(p\pi)}$; $0<p<1$

Dokaz izreka: $\displaystyle B(p,q)\Gamma(q+p) = \left(\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt\right)\left(\int_0^\infty x^{p+q-1} e^{-x} dx\right) = \int_0^\infty \left(\int_0^\infty x^{p+q-1} e^{-x} dx\right) \cdot \frac{t^{p-1}}{(1+t)^{p+q}} dt$

$$= \int_0^\infty \left(\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} x^{p+q-1} e^{-x} dx\right) dt = \int_0^\infty \left(\int_0^\infty \left(\frac{x}{1+t}\right)^{p+q-1} t^{p-1} e^{-x} \frac{dx}{1+t}\right) dt \qquad \frac{x}{1+t} = u \quad \frac{dx}{1+t} = du$$

$$= \int_0^\infty \left(\int_0^\infty u^{p+q-1} t^{p-1} e^{-u(1+t)} du\right) dt \overset{\text{Fub}}{=} \int_0^\infty \left(\int_0^\infty u^{p+q-1} t^{p-1} e^{-u} e^{-ut} dt\right) du = \int_0^\infty u^{q-1} e^{-u} \underbrace{\left(\int_0^\infty (ut)^{p-1} e^{-ut} u \, dt\right)}_{\Gamma(p)} du$$

$ut = v$

$$= \int_0^\infty u^{q-1} e^{-u} \left(\int_0^\infty v^{p-1} e^{-v} dv\right) du = \Gamma(p)\Gamma(q). \quad \blacksquare$$

Izrek (Stirlingova formula): $\displaystyle\lim_{s\to\infty} \frac{\Gamma(s+1)}{s^s e^{-s}\sqrt{2\pi s}} = 1$

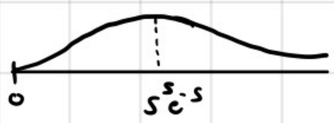
Posledica: $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$

Zgled: $\displaystyle\lim_{n\to\infty} \frac{(2n)!}{n^n 2^n n!} = \lim_{n\to\infty} \frac{(2n)!}{(2n)^{2n} e^{-2n}\sqrt{2\pi 2n}} \cdot \frac{(2n)^{2n} e^{-2n}\sqrt{2\pi 2n}}{n^n \cdot 2^n} \cdot \frac{n^n e^{-n}\sqrt{2\pi n}}{n!} \cdot \frac{1}{n^n e^{-n}\sqrt{2\pi n}}$

$$= \lim_{n\to\infty} \frac{2^{\cancel{2n}} \cancel{n^{2n}} \cancel{e^{-2n}} \sqrt{2\pi 2n}}{\cancel{n^n}\cancel{n^n} \cancel{2^n} \cancel{e^{-n}} \sqrt{2\pi n}} = \lim_{n\to\infty} \frac{2^n\sqrt{2}}{e^n} = 0$$

Dokaz izreka: $\Gamma(s+1) = \displaystyle\int_0^\infty x^s e^{-x} dx$ $\qquad \varphi(x) = x^s e^{-x}$

$\varphi'(x) = s x^{s-1} e^{-x} - x^s e^{-x} = e^{-x} x^{s-1}(s-x)$



Nova spremenljivka: $x = s(1+u)$, s tem val fiksiramo.

$$\Gamma(s+1) = \int_{-1}^\infty s^s (1+u)^s e^{-s} e^{-su} \cdot s\,du = s^s e^{-s} s \int_{-1}^\infty ((1+u)e^{-u})^s du$$

$$\lim_{s\to\infty} \frac{\Gamma(s+1)}{s^s e^{-s}\sqrt{s}} = \sqrt{s}\int_{-1}^\infty ((1+u)e^{-u})^s du \overset{?}{=} \sqrt{2\pi}$$

$u \mapsto (1+u)e^{-u}$, odvod: $e^{-u}(-u)$

$(1+u)e^{-u} \le 1$, zato pri velikih s ho že manjša. Glavni del integrala gre zato okoli 0, kjer je blizu 1.

$$\sqrt{s}\int_1^\infty ((1+u)e^{-u})^s du \le \sqrt{s}\, 2^{s-1} e^{-(s-1)} \int_1^\infty (1+u)e^{-u} du$$

$$\hookrightarrow \lim_{s\to\infty} \frac{\mu \cdot \sqrt{s} \cdot 2^{s-1}}{e^{s-1}} = 0$$

$$= \lim_{s\to\infty} \sqrt{s}\int_1^\infty ((1+u)e^{-u})^s du = 0$$

Videti moramo torej: $\displaystyle\lim_{s\to\infty} \sqrt{s}\int_{-1}^1 ((1+u)e^{-u})^s du = \sqrt{2\pi}$

$\psi(u) = (1+u)e^{-u}$

$(\ln \Psi)(u) = \ln(1+u) - u$

$(\ln \Psi)(u) = \left(u - \frac{u^2}{2} + \frac{u^3}{3} - \dots\right) - u = -\frac{u^2}{2} + u^3 \Phi(u)$; $\Phi(u) = \frac{(\ln \Psi)(u) + \frac{u^2}{2}}{u^3}$, dobře definováno na $u > -1$.

$(1+u)e^{-u} = e^{-\frac{u^2}{2}} e^{u^3 \Phi(u)}$

$\Rightarrow \lim_{s \to \infty} \sqrt{s} \int_{-1}^{1} e^{-\frac{su^2}{2}} e^{su^3 \Phi(u)} du$

$u\sqrt{s} = t$, $u = \frac{t}{\sqrt{s}}$

$\Rightarrow \lim_{s \to \infty} \sqrt{s} \int_{-\sqrt{s}}^{\sqrt{s}} e^{-\frac{t^2}{2}} e^{\frac{st^3}{\sqrt{s}^3} \Phi(\frac{t}{\sqrt{s}})} \frac{dt}{\sqrt{s}}$

$\Rightarrow \lim_{s \to \infty} \int_{-\sqrt{s}}^{\sqrt{s}} e^{-\frac{t^2}{2}} \cdot e^{\frac{t^3}{\sqrt{s}} \Phi(\frac{t}{\sqrt{s}})} dt$

$s = r^{10}$ $s \to \infty \Leftrightarrow r \to \infty$

$\lim_{r \to \infty} \int_{-r^5}^{r^5} e^{-\frac{t^2}{2}} \cdot e^{\frac{t^3}{r^5} \Phi(\frac{t}{r^5})} dt$

$\left| \int_{-r^5}^{-r} e^{-\frac{t^2}{2}} e^{\frac{t^3}{r^5} \Phi(\frac{t}{r^5})} dt \right| \leq r^5 e^{-\frac{r^2}{2}} \underbrace{e^{-\frac{1}{r^2} \Phi(-\frac{1}{r^4})}}_{1} \xrightarrow{r \to \infty} 0.$

Podobně $\left| \int_{r}^{r^5} e^{-\frac{t^2}{2}} e^{\frac{t^3}{r^5} \Phi(\frac{t}{r^5})} dt \right| \xrightarrow{r \to \infty} 0$

Ostatní tvoří lk: $[-r, r]$:

$\int_{-r}^{r} e^{-\frac{t^2}{2}} dt \cdot \underbrace{e^{-\frac{M}{r^2}}}_{\to 1} \leq \int_{-r}^{r} e^{-\frac{t^2}{2}} e^{\frac{t^3}{r^5} \Phi(\frac{t}{r^5})} dt \leq \int_{-r}^{r} e^{-\frac{t^2}{2}} dt \cdot \underbrace{e^{\frac{1}{r^2} M}}_{\to 1}$

$\downarrow r \to \infty$ $\qquad$ $-\frac{1}{r^4} < \frac{t}{r^5} < \frac{1}{r^4}$ $\qquad$ $\downarrow r \to \infty$

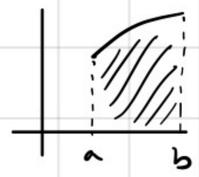
$\left|\Phi\left(\frac{t}{r^5}\right)\right| \leq M$ ~

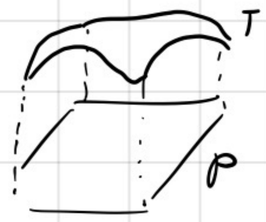
$\sqrt{2\pi}$ $\qquad$ $\sqrt{2\pi}$

$\Rightarrow$ Sledovaný výraz jen tedy proti $\sqrt{2\pi}$.

Riemannov integral v $\mathbb{R}^n$

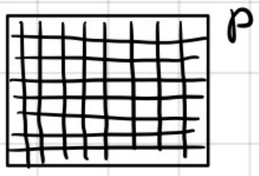
$n=1$:





$T = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in P,\ 0 \le z \le f(x,y)\}$

Kdaj obstaja volumen takega lika

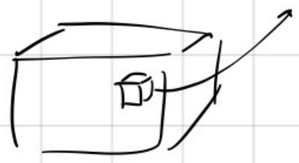


Delitev na pravokotnike, dobimo kriv kvader

Volumen telesa: $\sum_j f(\xi_j, \eta_j)\, Pl(P_j)$

Ali: $\rho(x,y,z)$, koliko je masa?

$m(K) = \sum \rho(x,y,z) \cdot V(K_j)$



Kvadri: $[a,b] = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid a_j \le x_j \le b_j\ ;\ j=1,\dots,n\}$

$V([a,b]) = \prod_1^n (b_j - a_j)$

Delitev kvadra K: $K = [a,b]$

D (delitev) dobimo z delitvami robov kvadra K:

$a_j = x_0^j < x_1^j < \dots < x_{m_j}^j = b_j$

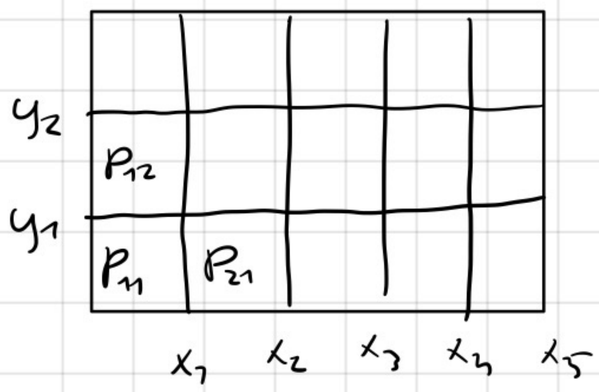
S temi smo kvader K razdelili na $m_1 \cdots m_n$ kvadrov

$[x_{i_1-1}^1, x_{i_1}^1] \times [x_{i_2-1}^2, x_{i_2}^2] \times \dots \times [x_{i_n-1}^n, x_{i_n}^n]$

$1 \le i_1 \le m_1 \qquad 1 \le i_2 \le m_2 \qquad 1 \le i_n \le m_n$

D = delitev je dana z delitvijo robov

D je sestavljena iz manjših kvadrov, ki jo porodi delitev robov



Definicija: Delitev D' kvadra K je finejša (oz. nadaljevanje) od delitve D, če vsebuje vse delilne točke delitve D (in morda še kakšno več), $D \le D'$

Opomba: ·) D delitev kvadra K: $\sum_{Q \in D} V(Q) = V(K)$

·) Če je D' finejša od D; → vsak kvader iz D' leži v enem od kvadrov iz D.

→ vsak kvader iz D lahko zapišemo kot nekaterih kvadrov iz D'.

Naj bo $f: K = [a,b] \subseteq \mathbb{R}^n \to \mathbb{R}$ omejena funkcija.

$m = m(f) = m(f,K) = \inf_K f(x)$

$M = M(f) = M(f,K) = \sup_K f(x)$

Naj bo D delitev kvadra K. Naj bo $Q \in D$.

$m(f,Q) = m(Q) = \inf_Q f(x)$

$M(f,Q) = M(Q) = \sup_Q f(x)$

Spodnja Darbouxjeva vsota funkcije f pri delitvi D: $s(f,D) = s(D) = \sum_{Q\in D} m(Q)V(Q)$

Zgornja Darbouxjeva vsota funkcije f pri delitvi D: $S(f,D) = S(D) = \sum_{Q\in D} M(Q)V(Q)$

Opomba: $m(K)V(K) \le s(f,D) \le S(f,D) \le M(K)V(K)$

Lema: Naj bo D' finejša od D. Tedaj je $s(D) \le s(D') \le S(D') \le S(D)$

Posledica: Naj bosta D_1, D_2 delitvi za K. Tedaj je $s(D_1) \le S(D_2)$

Dokaz posledice: $D' := D_1 \cup D_2$

Velja $D_1 \le D'$, $D_2 \le D'$.

Po lemi: $s(f,D_1) \le s(f,D') \le S(f,D') \le S(f,D_2)$.

Dokaz leme: Dovolj je pogledati primer, ko ima D' eno delilno točko več kot D.

Nekatere kvadre delitve D razdelimo na 2 nova kvadra delitve D'.

$Q \in D$, $Q = Q_1' \cup Q_2'$ $\quad Q_1', Q_2' \in D'$

i) $V(Q) = V(Q_1') + V(Q_2')$

ii) $m(Q) \le m(Q_j')$ $\quad M(Q) \ge M(Q_j')$

Ker za poljubni delitvi velja: $s(D_1) \le S(D_2) \Rightarrow \underbrace{\sup_D (s(f,D))}_{s(f)} \le \underbrace{\inf_D (S(f,D))}_{S(f)}$

Definicija: Funkcija f je na K integrabilna (po Darbouxju), če je $s(f) = S(f)$.

Če velja enakost, to vrednost trenutno označimo z I_D, sicer $\int_K f(x)dx = \int_K f(x)dV(x) = \underbrace{\int\dots\int_K}_{n\text{-krat}} f(x_1,\dots,x_n)dx_1dx_2\dots dx_n$.

Oznaka pri $n=2$: $\iint_P f(x,y)dxdy$ $\quad n=3$: $\iiint_K f(x,y,z)dxdydz$

Dvojni $\quad$ Trojni

$n=1$: $\int_{[a,b]} f(x)dx$

Riemannov integral:

Riemannova vsota: $K = [a,b]$ $\quad f: K \to \mathbb{R}$, ni nujno omejena

D delitev

$\forall Q \in D$ izberemo $\eta_Q \in Q$ (kvader → točka)

$$R(f,D,\eta) = \sum_{Q\in D} f(\eta_Q)V(Q)$$

Riemannova vsota funkcije f pri delitvi D in izboru točk $\eta = \{\eta_Q \in Q\}$

Če je D delitev, je $\Delta(D)$ maksimalna dolžina vseh dolžin vseh robov kvadrov delitve D.

Definicija: f je integrabilna po Riemannu na kvadru K, če obstaja limita njenih Riemannovih vsot.

$\lim_{\Delta(D)\to 0} R(f,R,\eta) = I_R$ → Riemannov integral.

To pomeni: $\forall \varepsilon > 0\ \exists \delta > 0: \forall D;\ \Delta(D) < \delta$ in vsak $\eta \Rightarrow |R(f,D,\eta) - I_R| < \varepsilon$

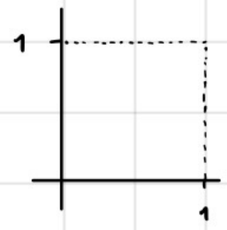
Zgled: $f(x) = a$

D delitev K, $m(Q) = M(Q) = f(\eta_Q) = a$

$\Rightarrow s(D) = S(D) = R(f,D,\eta_Q) = a \cdot V(K)$ $\quad$ r. $\int_K a\,dx = a \cdot V(K)$

Zgled: $K = [0,1] = \{(x,y)\in\mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1\} = [0,1]\times[0,1]$

$f(x,y) = \begin{cases} 1 & ; x,y \in \mathbb{Q} \\ 0 & \text{sicer} \end{cases}$



D delitev K: $m(Q) = 0 \Rightarrow s(f,D) = 0$

$M(Q) = 1 \Rightarrow S(f,D) = 1$ $\Rightarrow f$ ni integrabilna po Dar.

$R(f,D,\eta) = \sum_{P\in D} f(x_P,y_P)V(P)$

Če izberemo $x_P, y_P \in \mathbb{Q}$ $\forall P$, je $R(f,D,\eta) = 1$, če pa izberemo $(x_P,y_P) \notin \mathbb{Q}\times\mathbb{Q} \Rightarrow R(f,D,\eta) = 0$.

Opomba: Če ima $f: K \to \mathbb{R}$ limito Riemannovih vsot, je f omejena: $\lim_{\Delta(D)\to 0} R(f,D,\eta) = I_R$.

Avtomatično predpostavimo, da so funkcije omejene.

Izrek: Naj bo $f: K \to \mathbb{R}$ omejena. NTSE:

i) f je na K integrabilna po Darbouxu

ii) f je na K integrabilna po Riemannu

iii) $\forall \varepsilon > 0$ obstaja delitev D K: $S(D) - s(D) < \varepsilon$.

V tem primeru je $I_D = I_R$.

Trditev: Naj bo $f: K \to \mathbb{R}$ zvezna, potem je f na K integrabilna.

Dokaz trditve: Naj bo $f: K \to \mathbb{R}$ zvezna. Naj bo $\varepsilon > 0$. K je tudi kompaktna, zato je f enakomerno zvezna na K. Tedaj obstaja $\delta > 0$: za vsaki dve točki $x, x' \in K$, za kateri velja $\max\{|x_1 - x_1'|, \dots, |x_n - x_n'|\} < \delta$

$\Rightarrow |f(x) - f(x')| < \frac{\varepsilon}{V(K)}$

Naj bo D delitev K, da je $\Delta(D) < \delta$.

Poglejmo $S(f,D) - s(f,D) = \sum_{Q \in D} (M(Q) - m(Q))\, V(Q)$

Ker je f zvezna, zavzameno $M(Q)$ in $m(Q)$: obstaja x_Q: $f(x_Q) = M(Q)$

in x_Q': $f(x_Q') = m(Q)$.

$\Rightarrow f(x_Q) - f(x_Q') < \frac{\varepsilon}{V(K)}$

$$S(f,D) - s(f,D) = \sum_{Q \in K} (M(Q) - m(Q))\, V(Q) < \frac{\varepsilon}{V(K)} V(K) = \varepsilon. \quad \blacksquare$$

($n=1$: Če je zvezna na $[a,b]$ je integrabilna

Če je f zvezna na (a,b) in omejena, je integrabilna)

Dokaz izreka: (i $\Leftrightarrow$ iii)

$\Rightarrow$: $\sup_D s(f,D) = s(f) = I_D = S(f) = \inf_D S(f,D)$

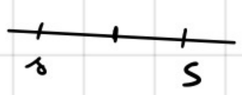
Naj bo $\varepsilon > 0$.

$\exists D_1$: $I_D - \frac{\varepsilon}{2} < s(f,D_1) \le I_D$

$\exists D_2$: $I_D \le S(f,D_2) < I_D + \frac{\varepsilon}{2}$

Naj bo $D := D_1 \cup D_2$

$I_D - \frac{\varepsilon}{2} \le s(f,D_1) \le s(f,D) \le I_D \le S(f,D) \le S(f,D_2) < I_D + \frac{\varepsilon}{2}$.



$\Leftarrow$: Naj bo $\varepsilon > 0$. $\exists D$ delitev: $S(f,D) - s(f,D) < \varepsilon$

$S(f) - s(f) < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow S(f) = s(f)$.

(i $\Leftrightarrow$ ii):

(ii $\Rightarrow$ iii): Obstaja $\lim_{\Delta(D)\to 0} R(f,D,\eta) = I_R$, $\eta = \{\eta_Q \in Q\}$

Naj bo $\varepsilon > 0$. Obstaja $\delta > 0$: $\forall D$, da $\Delta(D) < \delta$ in $\forall \eta$: $|R(f,D,\eta) - I_R| < \frac{\varepsilon}{3}$.

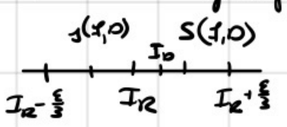
$R(f,D,\eta) = \sum_{Q \in D} M(Q) V(Q) = \sum_{Q \in D} \sup_Q f \cdot V(Q)$

Ker $\eta_Q \in Q$ lahko izbiramo poljubno, si z vrednostmi $f(\eta_Q)$ poljubno dobro približamo $M(Q) = \sup_Q f$.

To pomeni: $|S(f,D) - I_R| \le \frac{\varepsilon}{3}$

Podobno za spodnje vsote: $|s(f,D) - I_R| \le \frac{\varepsilon}{3}$

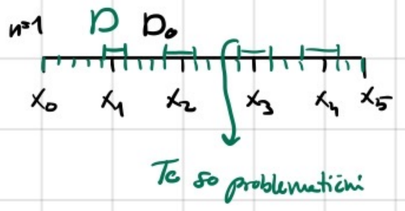
$\Rightarrow S(f,D) - s(f,D) \le \frac{2\varepsilon}{3} < \varepsilon \Rightarrow$ Integrabilna po Darbouxu.



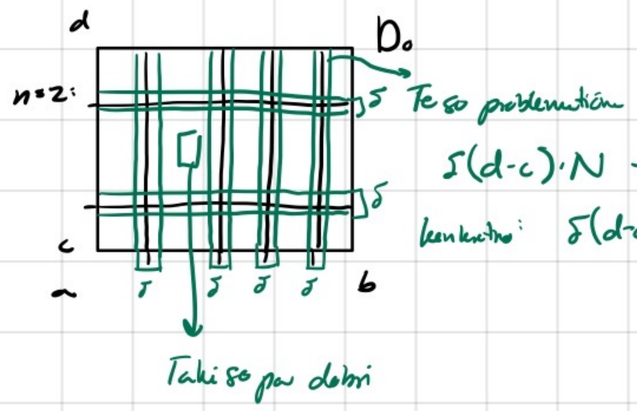
$\forall \varepsilon > 0 \Rightarrow I_R = I_D$

(i $\Rightarrow$ ii): Ce je D delitev: $s(f,D) \le R(f,D,\eta) \le S(f,D) \to$ Ni nujno, da velja za $\forall D$. Pri I_D je le eno dobro potrebna.

Pomožna trditev: Naj bo D_0 delitev kvadra K. Naj bo $\varepsilon>0$. Potem obstaja tak $\delta>0$ da za vsako delitev D, za katero je $s(D)<\delta$, velja: vsota prostornin kvadrov delitve D, ki niso vsebovani v kakšnem od kvadrov delitve D_0, manjša od ε.



$D_0: a=x_0<x_1\dots<x_N=b$, št. delilnih točk: N, $\delta\cdot N<\varepsilon$



$\delta(d-c)\cdot N+\delta(b-a)\cdot M$

konkretno: $\delta(d-c)\cdot 4+\delta(b-a)\cdot 2$. Pol: $\delta:=\dfrac{\varepsilon}{4(d-c)+2(b-a)}$

$n=3$:

Nadaljujmo dokaz ($i\Rightarrow ii$):

Vemo: $I_D=s=S$, obstaja. Naj bo $\varepsilon>0$. Potem obstaja delitev D_0, da velja

$$I_D-\frac{\varepsilon}{2}\le s(f,D_0)\le S(f,D_0)\le I_D+\frac{\varepsilon}{2}$$

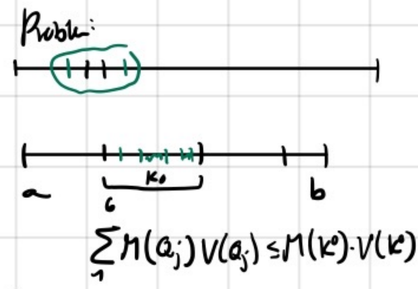
Uporabimo pomožno trditev: Obstaja $\delta>0$ da $\forall D$, za velja $s(D)<\delta$, je vsota prostornin kvadrov D, ki niso vsebovani v nobenem kvadru D_0, manjša od $\frac{\varepsilon}{2M}$.

f je omejena: $|f|\le M$ na K.

(Opomba: $R(f,D,\eta)=\sum_{Q\in D} f(\eta_Q)V(Q)$, en od faktorjev zagotavlja majhnost členov vsote)

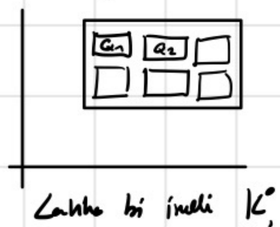
Sli je $|R(f,D,\eta)-I_D|<\varepsilon$ $\forall D$: $s(D)<\delta$ in $\forall\eta \Rightarrow \lim_{s(D)\to 0} R(f,D,\eta)=I_D$.

$$R(f,D,\eta)=\sum_{Q\in D} f(\eta_Q)V(Q)=\sum_{\substack{Q\subseteq K^\circ\\ K^\circ\in D_0}}{}' f(\eta_Q)V(Q)+\sum_{\text{ostali}}{}'' f(\eta_Q)V(Q)$$



$\sum_1 M(Q_j)V(Q_j)\le M(K^\circ)\cdot V(K^\circ)$

Ocenimo drugi kos: $\left|\sum_Q{}'' f(\eta_Q)V(Q)\right|\le\left|\sum_Q{}'' M\cdot V(Q)\right|<M\cdot\frac{\varepsilon}{2M}=\frac{\varepsilon}{2}$



Lahko bi imeli K°, ki ne vsebuje kakšnega Q. δ mora biti tako majhen, da vsak K° vsebuje vsaj en Q.

Pri kos: $\sum_{Q\subseteq K^\circ}{}' f(\eta_Q)V(Q)\le\sum_{Q\subseteq K^\circ}{}' M(Q)V(Q)\le S(f,D)=$

$$\sum_1^c M(Q_j)V(Q_j)\le\sum_1^c M(K^\circ)V(Q_j)=M(K^\circ)\sum_1^c V(Q_j)$$
$$\le M(K^\circ)\left(V(K^\circ)-E(K^\circ)\right)$$

$$\sum_{Q\subseteq K^\circ}{}' M(Q)V(Q)\le\sum_{K^\circ} M(K^\circ)\left(V(K^\circ)-E(K^\circ)\right)=S(f,D_0)-\sum_{K^\circ}M(K^\circ)E(K^\circ)\le S(f,D_0)+\frac{\varepsilon}{2}$$

in $\sum_{K^\circ}E(K^\circ)M(K^\circ)\le M\cdot\sum E(K^\circ)\le M\cdot\frac{\varepsilon}{2M}=\frac{\varepsilon}{2}$

Podobno za narobe

Na koncu: $|R(f,D,\eta)-I_D|<\varepsilon$. ■

I_D obstaja, izb. $\varepsilon>0$. $\exists D_0$: $I_D-\frac{\varepsilon}{2}<s(f,D_0)\le S(f,D_0)<I_D+\frac{\varepsilon}{2}$

$\exists\delta>0$: (pomožna) in $|f|\le M$.

D: $s(D)<\delta$

$R(f,D,\eta)=\sum_{Q\subseteq K^\circ}{}'+\sum{}''$

$\sum{}''<\frac{\varepsilon}{2}$ (pomožna in omejenost f)

$\sum_{Q\subseteq K^\circ}{}' f(\eta_Q)V(Q)\le S(f,D_0)-\underbrace{\sum_{K^\circ\in D_0}M(K^\circ)E(K^\circ)}_{\ge\varepsilon/2}\Rightarrow\sum{}'\le S(f,D_0)+\frac{\varepsilon}{2}<I_D+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$

$I_D-\varepsilon<\sum{}' f(\eta_Q)V(Q)<I_D+\varepsilon$

$\Rightarrow$ $I_D-\frac{3\varepsilon}{2}<R(f,D,\eta)<I_D+\frac{3\varepsilon}{2}$

Osnovne lastnosti Riemannovega integrala po kvadrih

$K \subseteq \mathbb{R}^n$, kvader, f, g integrabilni na K.

1.) Naj bosta $\lambda, \mu \in \mathbb{R}$. Tedaj je tudi $\lambda f + \mu g$ integrabilna na K.

in: $\int_K (\lambda f(x) + \mu g(x))\,dx = \lambda \int_K f(x)\,dx + \mu \int_K g(x)\,dx$.

Torej: množica integrabilnih funkcij na K je vektorski prostor nad $\mathbb{R}$, integral pa je linearen funkcional na tem prostoru.

Tega ni zapiskih na roke, niso pa presenetljive lastnosti.

Dokaz: Z Riemannovimi vsotami: limita vsote je vsota limit...

Zakaj Darbouxa ni dovolj? $\sup(f+g) \leq \sup f + \sup g$ $\quad (f = x^2, g = -x^2$ na $[-1,1])$

2.) Če je $f(x) \leq g(x)$ $\forall x \in K$, je $\int_K f(x)\,dx \leq \int_K g(x)\,dx$

Dokaz: Riemannove ali Dar. vsote

3.) Tudi $|f|$ je integrabilna: $\left|\int_K f(x)\,dx\right| \leq \int_K |f(x)|\,dx$

Dokaz: Če je $|f|$ integrabilna, sledi neenakost iz (2.). $\quad -f(x) \leq f(x) \leq |f(x)|$

Zakaj je $|f|$ integrabilna?

Trdimo: Če je D delitev K, je $S(|f|, D) - s(|f|, D) \leq S(f, D) - s(f, D)$. $\quad \sum (M(Q) - m(Q))\, V(Q)$

Če to velja, je $|f|$ integrabilna

$Q \in K$: $\sup_Q |f| - \inf_Q |f| \leq \sup_Q f - \inf_Q f$

i) $f(x) \geq 0$ na Q: $\sup_Q |f| = \sup_Q f$, $\inf_Q |f| = \inf_Q f$ $\Rightarrow$ $\sup_Q |f| - \inf_Q |f| \leq \sup_Q f - \inf_Q f$.

ii) $f(x) \leq 0$ na Q: $\sup_Q |f| = \sup_Q -f = -\inf_Q f$, $\inf_Q |f| = \inf_Q -f = -\sup_Q f$ $\Rightarrow$ spet dobimo enačaj.

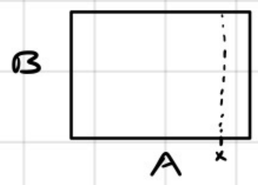
iii) f zavzame poz. in neg. vrednosti na Q:

$\sup_Q |f| = \max\{\sup_Q f, -\inf_Q f\}$

$\inf_Q |f| \geq 0$, zato $\sup_Q |f| - \inf_Q |f| \leq \sup_Q |f| = \max\{\sup_Q f, -\inf_Q f\} \leq \sup_Q f + (-\inf_Q f)$

Fubinijev izrek:

Izrek (1.): Naj bo $A \subseteq \mathbb{R}^n$ kvader in $B \subseteq \mathbb{R}^m$ kvader. Naj bo $f: A \times B \subseteq \mathbb{R}^{n+m} \to \mathbb{R}$ integrabilna. Naj bo $\forall x \in A$ funkcija $y \mapsto f(x,y)$ integrabilna na B. Potem je funkcija $x \mapsto \int_B f(x,y)\,dy$ integrabilna na A in velja: $\iint_{A \times B} f(x,y)\,dx\,dy = \int_A \left(\int_B f(x,y)\,dy \right) dx$.



(2.): Naj bosta $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ kvadra, $f: A \times B \to \mathbb{R}$ integrabilna. Naj bo $\forall y \in B$, naj bo funkcija $x \mapsto f(x,y)$ integrabilna na A. Potem je funkcija $y \mapsto \int_A f(x,y)\,dx$ integrabilna na B in velja:

$$\iint_{A \times B} f(x,y)\,dx\,dy = \int_B \left(\int_A f(x,y)\,dx \right) dy$$

Posledica: Če je f zvezna na $A \times B$, potem $\iint_{A \times B} f(x,y)\,dx\,dy = \int_A \left(\int_B f(x,y)\,dy \right) dx = \int_B \left(\int_A f(x,y)\,dx \right) dy$.

Posledica: $f: [a,b] \times [c,d] \to \mathbb{R}$ zvezna: tedaj je

$$\iint_{[a,b] \times [c,d]} f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Posledica: $f: \overbrace{[a,b] \times [c,d] \times [g,h]}^{K} \to \mathbb{R}$ zvezna:

$$\iiint_K f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_c^d \left(\int_g^h f(x,y,z)\,dz \right) dy \right) dx = 5 \text{ dvojnih formul.}$$

Zgled: $P = [-1,2]\times[0,1]$

$$\iint_P (x^2y+y^2+2)\,dx\,dy = \int_{-1}^{2}\left(\int_0^1 x^2y+y^2+2\,dy\right)dx = \int_{-1}^{2} \tfrac{1}{2}x^2y^2+\tfrac{1}{3}y^3+2y\Big|_0^1\,dx = \int_{-1}^{2} \tfrac{1}{2}x^2+\tfrac{1}{3}+2\,dx = \tfrac{x^3}{6}+\tfrac{7}{6}x\Big|_{-1}^{2} = \tfrac{3}{2}+7 = \underline{\underline{\tfrac{17}{2}}}$$

Enako: $\int_0^1\left(\int_{-1}^{2} x^2y+y^2+2\,dx\right)dy$

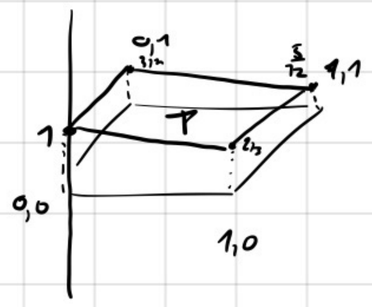
Zgled: $K = [0,1]\times[-1,1]\times[0,1]$ $\quad f(x,y,z) = x+y^2+z^3$ $\quad$ (oznaka: $\int_0^1 dx\int_{-1}^1 dy\int_0^1 (x+y^2+z^3)\,dz$)

$$\iiint_K x+y^2+z^3\,dx\,dy\,dz = \int_0^1\left(\int_{-1}^1\left(\int_0^1 x+y^2+z^3\,dz\right)dy\right)dx = \int_0^1\left(\int_{-1}^1 x+y^2+\tfrac{1}{4}\,dy\right)dx = \int_0^1 2x+\tfrac{2}{3}+\tfrac{1}{2}\,dx = x^2+\tfrac{7}{6}x\Big|_0^1 = \underline{\underline{\tfrac{13}{6}}}$$

Zgled: $f(x,y) = 1-\frac{1}{3}x-\frac{1}{4}y$ $\quad P = [0,1]\times[0,1]$

$T = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in P,\ 0\le z\le f(x,y)\}$

$V(T) = ?$



$$V(T) = \iint_P f(x,y)\,dx\,dy = \int_0^1\int_0^1\left(1-\tfrac{1}{3}x-\tfrac{1}{4}y\right)dy\,dx = \int_0^1\left(1-\tfrac{1}{3}x-\tfrac{1}{8}\right)dx = \int_0^1 \tfrac{7}{8}-\tfrac{1}{3}x\,dx = \underline{\underline{\tfrac{17}{24}}}$$

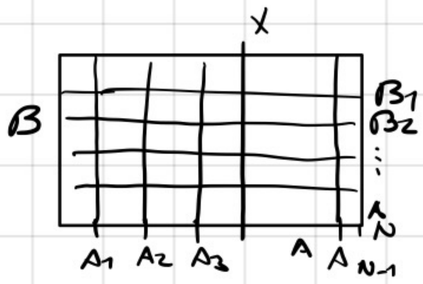
Dokaz Fubinijevega izreka:

$A\subseteq\mathbb{R}^n$ (x)

$B\subseteq\mathbb{R}^m$ (y)

$A\times B\subseteq\mathbb{R}^n\times\mathbb{R}^m$ $\quad$ za $x\in A$: $f(x,y) = f^x(y)$, $f^x: B\to\mathbb{R}$

$g(x) = \int_B f^x(y)\,dy = \int_B f(x,y)\,dy$ – To obstaja po predpostavki.



D_A ... delitev A: $A_1,\dots,A_N$

D_B ... delitev B: $B_1,\dots,B_M$

$D := D_A\times D_B$, $\quad K_{ij} := A_i\times B_j$

$x\in A_i$

$$\underset{\substack{\cap\\ m_{ij}}}{\inf_{A_i\times B_j} f} \le \inf_{B_j}\left(f^x(y)\right) \le \sup_{B_j}\left(f^x(y)\right) \le \underset{\substack{\cap\\ M_{ij}}}{\sup_{A_i\times B_j} f} \qquad / \cdot V(B_j),\ +\text{ seštejemo po } j$$

$$\Rightarrow \sum_j m_{ij}\cdot V(B_j) \le s(f^x, D_B) \le S(f^x, D_B) \le \sum_j M_{ij}\,V(B_j)$$

Integral leži vmes – g.

Integral $g(x) = \int_B f^x(y)\,dy$ obstaja $\forall x \Rightarrow \sum_j m_{ij}\,V(B_j) \le g(x) \le \sum_j M_{ij}\cdot V(B_j)$ $\quad \forall x\in A_i$

$$\Rightarrow \sum_j m_{ij}\,V(B_j) \le \inf_{A_i} g \le \sup_{A_i} g \le \sum_j M_{ij}\,V(B_j) \qquad / \ V(A_i),\ \text{seštejemo po } i$$

$$\Rightarrow \sum_{ij} m_{ij}\,V(A_i)_n V(B_j)_m \le s(g, D_A) \le S(g, D_A) \le \sum_{ij} M_{ij}\,\underbrace{V(A_i)_n\,V(B_j)_m}_{V(A_i\times B_j)}$$

$$\Rightarrow s(f, D_A\times D_B) \le s(g, D_A) \le S(g, D_A) \le S(f, D_A\times D_B)$$

Ker je f integrabilna na produktu:

$\Rightarrow \forall\varepsilon>0\ \exists D_A\times D_B$ delitev $A\times B$, da je razlika $S(f, D_A\times D_B) - s(f, D_A\times D_B) < \varepsilon$

$\Rightarrow S(g, D_A) - s(g, D_A) < \varepsilon \Rightarrow g$ je integrabilna na A.

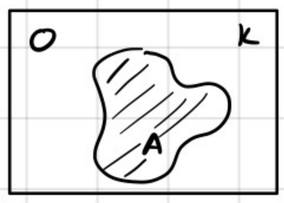
Integrala sta enaka:

$$\iint_{A\times B} f(x,y)\,dx\,dy = \int_A g(x)\,dx = \int_A\left(\int_B f(x,y)\,dy\right)dx \qquad \blacksquare$$

Naj bo A omejena $\subseteq\mathbb{R}^n$, $f: A\to\mathbb{R}$ omejena.

Kaj je $\int_A f(x)\,dx$? Kaj je $V(A)$ – prostornina?

Ker je A omejena obstaja K kvader, da je $A \subseteq K$.

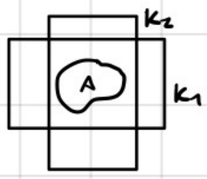


Definiramo $\tilde{f} = \begin{cases} f(x) & ; x \in A \\ 0 & ; x \notin A \end{cases}$

Definicija: Omejena funkcija $f: A \to \mathbb{R}$ je integrabilna na A, če je $\tilde{f}$ integrabilna na $K \supseteq A$ (kvader).

Tedaj $\int_A f(x)\,dx = \int_K \tilde{f}(x)\,dx$

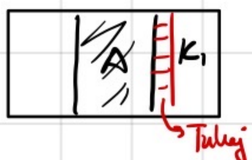
Opomba:



Naj bosta K_1, K_2 kvadra, ki vsebujeta A. Tedaj je $\tilde{f}$ int. na $K_1 \Leftrightarrow \tilde{f}$ je int. na K_2. $\int_{K_1} \tilde{f}(x)\,dx = \int_{K_2} \tilde{f}(x)\,dx$.

Tedaj je definicija integrabilnosti in vrednosti integrala dobra.

Opomba: Če je A že kvader in ga damo v rečjega, a dobimo enak rezultat.



Tukaj smo lahko nekaj dodali - malo je ker so majhni volumni

Zgled: $A = [0,1]^2 \cap \mathbb{Q}^2$

$f(x,y) = 1$ je zvezna na A. Ali je integrabilna na A?

$\tilde{f}(x,y) = \begin{cases} 1 & ; x,y \in \mathbb{Q} \\ 0 & \text{sicer} \end{cases}$

$\tilde{f}$ ni integrabilna na $[0,1]^2 \supseteq A$.

f ni integrabilna na A.

$G(x,y) = \begin{cases} 1 & ; x=y=1 \\ 0 & ; \text{sicer} \end{cases}$ je integrabilna na A in $\iint_A G(x,y)\,dx\,dy = 0$

Trditev: Naj bo A omejena podmnožica, $A \subseteq \mathbb{R}^n$. Naj bosta $f, g: A \to \mathbb{R}$ integrabilni na A in naj bosta $\lambda, \mu \in \mathbb{R}$. Tedaj je $\lambda f + \mu g$ integrabilna na A. $\int_A (\lambda f + \mu g)(x)\,dx = \lambda \int_A f(x)\,dx + \mu \int_A g(x)\,dx$.

Množica integrabilnih funkcij na A tvori vektorski prostor nad $\mathbb{R}$.

Integral je linearen funkcional na tem prostoru.

Dokaz: Velja, ker trditev velja za $A \subseteq$ kvader in $\widetilde{\lambda f + \mu g} = \lambda \tilde{f} + \mu \tilde{g}$.

$$\int_K \widetilde{(\lambda f + \mu g)}(x)\,dx = \lambda \int_K \tilde{f}(x)\,dx + \mu \int_K \tilde{g}(x)\,dx \quad (\text{so ravno } I_A)$$

Kolikšna pa je prostornina $A \subseteq \mathbb{R}^n$? $V(A) = ?$

$\chi_A(x) = \begin{cases} 1 & ; x \in A \\ 0 & ; \text{sicer} \end{cases}$ Karakteristična funkcija množice A.

Za $f(x) = 1$ na A, je $\tilde{f}(x) = \chi_A(x)$.

Definicija: Omejena množica $A \subseteq \mathbb{R}^n$ ima prostornino, (Jordanovo prostornino) če je funkcija 1 integrabilna na A.

Tedaj je $V(A) = \int_A 1\,dx$

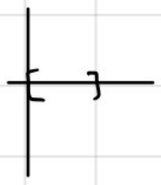
$A \subseteq K$

$\tilde{1} = \chi_A$: $V(A) = \int_A 1\,dx = \int_K \chi_A(x)\,dx$

Opomba: Če ima A prostornino, so vse konstantne funkcije integrabilne na A:

$$\int_A \lambda\,dx = \lambda \cdot V(A)$$

Zgled: $A = [0,1]^2 \cap \mathbb{Q}^2$ nima ploščine.

Opomba: 

$A = \{(x,0);\ x \in [0,1]\}$

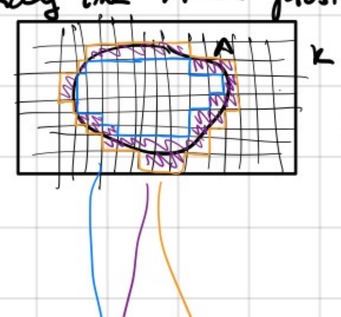
$V_1(A) = 1 \quad V_2(A) = 0$

dim dim

Opomba: $f: A \to \mathbb{R}$.

$f: \mathbb{R}^n \to \mathbb{R} \Longrightarrow \tilde{f} = f \cdot \chi_A$

Kdaj ima $A \subseteq \mathbb{R}^n$ prostornino? $\int_A 1\,dx = \int_K \chi_A(x)\,dx$ - Kdaj to obstaja?



D ... delitev K

$S(\chi_A, D) = \sum_{Q \in D} M_Q V(Q) \qquad s(\chi_A, D) = \sum_{Q \in D} m_Q V(Q)$

$M_Q = \sup_Q \chi_A = \begin{cases} 0 & Q \cap A = \emptyset \\ 1; & Q \cap A \neq \emptyset \end{cases}$

$m_Q = \inf_Q \chi_A = \begin{cases} 0 & \text{sicer} \\ 1 & Q \subseteq A \end{cases}$

$S(\chi_A, D)$ = vsota prostornin kvadrov $Q \in D$, $Q \cap A \neq \emptyset$

$s(\chi_A, D)$ = $Q \in D$, $Q \subseteq A$.

Integral $\int_K \chi_A\,dx$ obstaja $\Leftrightarrow \forall \varepsilon > 0\ \exists D$ delitek K: $S(\chi_A, D) - s(\chi_A, D) < \varepsilon$.

= vsota prostornin kvadrov $Q \in D$: $Q \cap A \neq \emptyset$ in $Q \cap A^c \neq \emptyset$.

Kvadri so povezani!

Ker so Q povezane množice $\Rightarrow$ vsak Q: $Q \cap A \neq \emptyset$ in $Q \cap A^c \neq \emptyset$

$\Rightarrow Q$ seka $\partial A \Leftrightarrow Q \cap \partial A \neq \emptyset$.

Trditev: Omejena množica A v $\mathbb{R}^n$ ima prostornino $\Leftrightarrow V(\partial A) = 0$.

Zgled: $A = [0,1]^2 \cap \mathbb{Q}^2$

$\partial A = [0,1]^2 \quad V(\partial A) = 1 \neq 0$.

$B \subseteq \mathbb{R}^n$ omejena. $V(B) = 0$?

Če obstaja: $V(B) = \sup_D (s(\chi_B, D)) = \inf_D (S(\chi_B, D))$

$V(B) = 0 \Leftrightarrow \inf_D S(\chi_B, D) = 0$

$S(\chi_B, D)$ = vsota prostornin kvadrov $Q \in D$: $B \cap Q \neq \emptyset$.

Dokaz trditve: $\Leftarrow$: $V(\partial B) = 0 \Rightarrow \forall \varepsilon > 0\ \exists D$ delitev: $S(\chi_{\partial A}, D) < \varepsilon$

$\Rightarrow$ vsota prostornin kvadrov $Q \in D$, $Q \cap \partial A \neq \emptyset$. Takoj jih je nekaj reč.

$S(\chi_A, D) - s(\chi_A, D)$ = vsota prostornin $Q \in D$: $Q \cap A \neq \emptyset$ in $Q \cap A^c \neq \emptyset$. (tak Q im presek z ∂A)

Členi ke vsoti so vsebovani v vsoti. $\Rightarrow S(\chi_A, D) - s(\chi_A, D) < \varepsilon$ (χ_A je integrabilna in A ima prostornino)

B omejena, $V(B) = 0 \Leftrightarrow \inf_D S(\chi_B, D) = 0 \Leftrightarrow \int_B 1\,dx = \int_K \chi_B\,dx$

$\sup_n (\chi_B, D) = 0$

$\inf_D s(\chi_B, D) \Rightarrow \forall D\ s(\chi_B, D) = 0 \Rightarrow \operatorname{Int} B = \emptyset$.

$= \sum_{\substack{Q \in D \\ Q \subseteq B}} V(Q)$

Trditev: Naj bo B omejena $\subseteq \mathbb{R}^n$. Velja: $V(B) = 0 \Leftrightarrow \forall \varepsilon > 0\ \exists$ kvadri $Q_1, \dots, Q_n$:

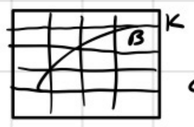
i) $B \subseteq \bigcup_i^n Q_i$

ii) $\sum_{i=1}^n V(Q_i) < \varepsilon$

Opombe:
- $Q_1, \dots, Q_n$ kvadri v $\mathbb{R}^n$. Naj bo $A := \bigcup_1^n Q_i$. Potem obstaja $V(A)$.
- $V(A) \le \sum_{j=1}^n V(Q_j)$ (ker se nekateri prekrivajo)
- $B \subseteq C$, oba omejeni, oba $\subseteq K$. Za delitev D: $S(\chi_B, D) \le S(\chi_C, D)$

$$\inf_D S(\chi_B, D) \le \inf_D S(\chi_C, D)$$

Dokaz trditve: ($\Rightarrow$): Naj bo $V(B) = 0$. $B \subseteq K$, kvader in naj bo $\varepsilon > 0 \Rightarrow \exists D$ delitev: $S(\chi_B, D) < \varepsilon$ (ii)



$$S(\chi_B, D) = \sum_{Q \cap B \neq \emptyset} V(Q)$$

i) $B \subseteq \bigcup_{Q \cap B \neq \emptyset} Q$ ✓

($\Leftarrow$): Naj bo $\varepsilon > 0$. Obstajajo $Q_1, \dots, Q_n$ kvadri: $B \subseteq \bigcup_j Q_j$ (C) in $\sum V(Q_j) < \varepsilon$.

$$\inf_D S(\chi_B, D) \le \inf_D S(\chi_C, D) = V(C) \le \sum V(Q_j) < \varepsilon$$

$$\Rightarrow \forall \varepsilon : \inf S(\chi_B, D) < \varepsilon \Rightarrow \inf_D S(\chi_B, D) = 0 \Rightarrow V(B) = 0.$$

Trditev: Naj bodo $B_1, \dots, B_N \subseteq \mathbb{R}^n$ množice s prostornino 0. Potem $V(\bigcup B_i) = 0$.

Dokaz: $B := \bigcup_1^N B_j$. Vemo: $V(B_j) = 0$, $j = 1, \dots, N$.

Naj bo $\varepsilon > 0$: $\forall j = 1, \dots, N\ \exists Q_1^j, \dots, Q_{N_j}^j$ kvadri da je: $B_j \subseteq \bigcup_{i=1}^{N_j} Q_i^j$ in $\sum_{i=1}^{N_j} V(Q_i^j) < \frac{\varepsilon}{N}$

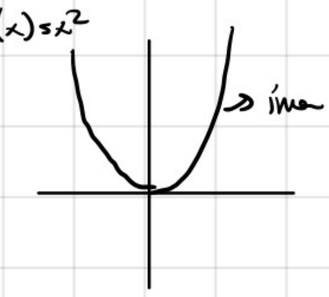
$$\Rightarrow B \subseteq \bigcup_{j=1}^N \bigcup_{i=1}^{N_j} Q_i^j \text{ in } \sum_j \sum_i V(Q_i^j) \le \sum_1^N \frac{\varepsilon}{N} = \varepsilon \quad \blacksquare$$

Trditev: Naj bo $f: K \to \mathbb{R}$ integrabilna, K pa kvader v $\mathbb{R}^n$.

Naj bo $\Gamma_f = \{(x, f(x)) \in K \times \mathbb{R}^n \subseteq \mathbb{R}^{n+1}\}$

Tedaj je $V_{n+1}(\Gamma_f) = 0$.

Zgled: $f(x) = x^2$



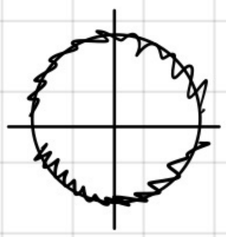
$\Rightarrow$ ima ploščino 0.

Zgled: $R > 0$ $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = R\} =: A$. Obstaja $V_2(A)$.

$V_2(A)$ obstaja $\Leftrightarrow V_2(\partial A) = 0$.

∂A je unija dveh grafov: $\partial A = \Gamma(\sqrt{R^2 - x^2}) \cup \Gamma(-\sqrt{R^2 - x^2})$, ki sta grafa nad $[-R, R]$.

(integrabilni funkciji)



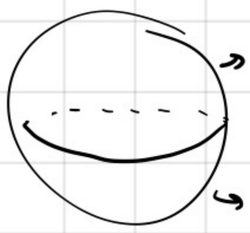
$$V(\Gamma(\sqrt{R^2 - x^2})) = V(\Gamma(-\sqrt{R^2 - x^2})) = 0$$

$\Rightarrow$ Unija tudi 0

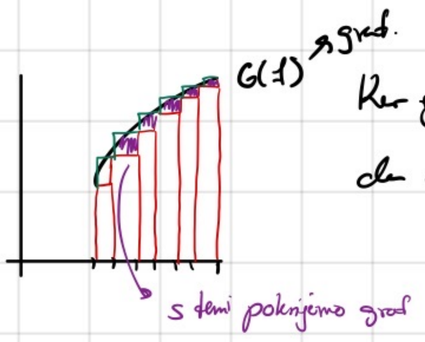
$\Rightarrow V(\partial A) = 0.$

Opomba: Trditev velja tudi za grafe nad drugimi kodimenzije 1 podprostori.

Zgled: Krogla



Dokaz trditve:



Ker je f integrabilna na K za $\varepsilon > 0$ obstaja tak delitev D kvadra K, da je $S(f, D) - s(f, D) < \varepsilon$

$$S(f, D) - s(f, D) = \sum_{Q \in D} (M(Q) - m(Q)) V_n(Q) = \sum_{Q \in D} V_{n+1}(\underbrace{Q \times [m(Q), M(Q)]}_{K_Q,\ \text{kvadri v } \mathbb{R}^{n+1}}) < \varepsilon$$

Ampak $G(f) \subseteq \bigcup_{Q \in D} K_Q$ $\blacksquare$

Trditev: Naj bo $A \subseteq \mathbb{R}^n$ množica s prostornino (A je avtomatično omejena). Naj bo $A \subseteq K$ kvader. Tedaj ima $K \setminus A$ tudi prostornino in $V(A) + V(K-A) = V(K)$

Dokaz: χ_A je integrabilna na K, 1 je tudi int. na $K \Rightarrow 1-\chi_A$ je integrabilna na K. $1-\chi_A = \chi_{K\setminus A}$, zato je χ_{K-A} je int. na $K \Rightarrow$ obstaja $V(K-A)$ in ker $1-\chi_A + \chi_{K-A} \Rightarrow \int_K 1\,dx = \int_K \chi_A dx + \int_K \chi_{K-A} dx \Rightarrow V(K) = V(A) + V(K\setminus A)$. ■

Trditev: Naj bo $A \subseteq K$ množica s prostornino 0: $V(A)=0$. Naj bo $f: K \to \mathbb{R}$ omejena funkcija, kjer velja $f(x)=0$, $x \notin A$.
Tedaj je f integrabilna na K in $\int_K f(x)dx = 0$.

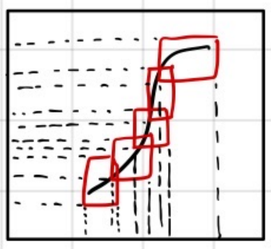
(Ekvivalentno: A, $V(A)=0$, $f: A \to \mathbb{R}$ omejena $\Rightarrow \exists \int_A f(x)dx = 0$)

Posledica: Naj bo F integrabilna na K in $G: K \to \mathbb{R}$ omejena. Če $V(\{x \in K \mid F(x) \neq G(x)\}) = 0$, je tudi G integrabilna na K in
$$\int_K G(x)dx = \int_K F(x)dx$$

Opomba: Če spremenimo nekaj točk, se funkcija še vedno integrabilni, tudi vrednost se ne spremeni.
Integral je neobčutljiv na omejene spremembe na množici s prostornino 0.

Dokaz posledice: $G-F = f$, f je omejena in f je 0 za $x \in \{x \in K \mid F(x) = G(x)\}$
Drugje ni $0 \Rightarrow f$ je int. in $\int_K f(x)dx = 0 \Rightarrow G = F + f \Rightarrow G$ je int. na K in
$$\int_K G(x)dx = \int_K F(x)dx + \overset{0}{\int_K f(x)dx}$$ ■

Dokaz trditve: f je omejena: $|f| \leq M \overset{\neq 0}{<} \infty$



$V(A) = 0$. Naj bo $\varepsilon > 0$. $\exists$ kvadri $Q_1, \dots, Q_N$, da je $A \subseteq \bigcup_1^N Q_j$ in $\sum_{j=1}^N V(Q_j) < \frac{\varepsilon}{2M}$
Kvadri $Q_1, \dots, Q_N$ porodijo delitev kvadra K s projekcijami njihovih oglišč na robove $K =: D$
BŠS: A leži v notranjosti $\bigcup_1^N Q_j$

Zapišem $S(f,D) = \sum_{R \in D} M(R) \cdot V(R) = \sum_{\substack{R \subseteq \bigcup_1^N Q_j \\ R \in D}} M(R) V(R) \leq M \cdot \sum_{\substack{R \in D \\ R \subseteq \bigcup_1^N Q_j}} V(R) \leq M \cdot \sum_1^N V(Q_j) = M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}$

(zunaj $\bigcup_1^N Q_j$ je $f \equiv 0$)

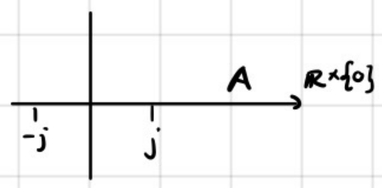
Torej $S(f,D) < \frac{\varepsilon}{2}$.
Podobno je $s(f,D) > -\frac{\varepsilon}{2}$
$\Rightarrow S(f,D) - s(f,D) < \varepsilon$
Integral je vmes: $\int_K f(x)dx = 0$.

Naj bo $A \subseteq \mathbb{R}^n$. A NI NUJNO omejena

Definicija: Množica A ima mero enako 0: $m(A) = 0$, če za vsak $\varepsilon > 0$ obstajajo $Q_1, \dots$ največ števno mnogo kvadrov, da velja:
i) $A \subseteq \bigcup_1^\infty Q_j$
ii) $\sum_1^\infty V(Q_j) < \varepsilon$.

Posledica: Če je $V(A) = 0 \Rightarrow m(A) = 0$.

Zgled: $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$



Naj bo $\varepsilon > 0$.
$Q_j := (-j, j) \times \left[0, \frac{\varepsilon}{2j \cdot 2^{j+2}}\right]$
$V_2(Q_j) = 2j \cdot \frac{\varepsilon}{2j \cdot 2^{j+2}} = \frac{\varepsilon}{2^{j+2}}$
$A \subseteq \bigcup_1^\infty Q_j$

$$\sum_1^\infty V(Q_j) = \sum_{j=1}^\infty \frac{\varepsilon}{2^{j+2}} = \frac{\varepsilon}{2^2} < \underline{\underline{\varepsilon}}$$

$\Rightarrow$ množica ima mero $\underline{\underline{m(A)=0}}$.

Zgled: Omejena, nima ploščine, ima mero.

$A = \mathbb{Q}$, $A \subseteq \mathbb{R}$.

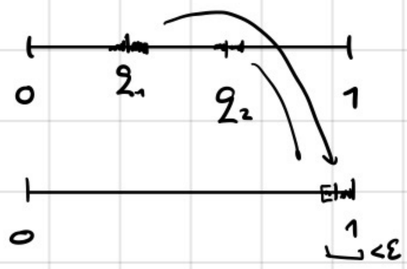
$A \cap [0,1]$ je omejena $\Rightarrow m(A)=0$

$A = \{q_1, \dots, q_n, \dots\}$, kar je števna množica.

Naj bo $\varepsilon > 0$. $Q_j = \left[q_j - \frac{\varepsilon}{2 \cdot 2^{j+1}}, q_j + \frac{\varepsilon}{2 \cdot 2^{j+1}}\right]$

i) $A \subseteq \bigcup_1^\infty Q_j$

ii) $\sum_1^\infty V_1(Q_j) = \sum_1^\infty \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} < \varepsilon$ ✓



(Racionalnih števil je malo)

Trditev: Naj bodo $A_1, \dots$ množice z mero 0 v $\mathbb{R}^n$: $m(A_j)=0$ $\forall j$.

Tedaj je $m(\bigcup A_j) = 0$ (števna unija ima še vedno mero 0).

Posledica: Vsaka števna množica v $\mathbb{R}^n$ ima mero enako 0.

(ker je števna unija točk, ki imajo prostornino 0)

$A = [0,1]^2 \cap \mathbb{Q}^2$ ne obstaja $V(A)$, vemo pa $m(A)=0$.

Dokaz trditve: $\forall i$ $m(A_i)=0$. Naj bo $\varepsilon > 0$. $\forall i$ $\exists Q_1^i, Q_2^i, \dots$

i) $A_i \subseteq \bigcup_{j=1}^\infty Q_j^i$

ii) $\sum_{j=1}^\infty Q_j^i < \frac{\varepsilon}{2^{i+1}}$

$$\Rightarrow \bigcup_1^\infty A_i \subseteq \bigcup_i \bigcup_j Q_j^i$$

$$\sum_i \sum_j V(Q_j^i) \leq \sum_i \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2} < \underline{\underline{\varepsilon}}$$

■

Trditev: Naj bo $f: \mathbb{R}^n \to \mathbb{R}$ zv., tedaj je $m_{n+1}(G(f)) = 0$.

Dokaz: $\mathbb{R}^n = \bigcup_{j=1}^\infty [-j,j]^n$, f je integrabilna na Q_j $\forall j$, $Q_j = [-j,j]^n$

$\Rightarrow V(G(f)|_{Q_j}) = 0 \Rightarrow m(G(f)|_{Q_j}) = 0$

$G(f) = \bigcup_{j=1}^\infty G(f)|_{Q_j}$, števna unija množic z mero 0 ima mero 0: $m(G(f)) = 0$.

Opomba: definicija množic z mero 0 je lahko podana tudi z odprtimi kvadri:

$A \subseteq \mathbb{R}^n$, $m(A)=0 \Leftrightarrow \forall \varepsilon > 0$ $\exists Q_1, \dots$ i) $A \subseteq \bigcup_i \operatorname{Int}(Q_i)$

ii) $\sum_j^\infty V(\operatorname{Int}(Q_j)) = \sum_j^\infty V(Q_j)$

iz definicije z odprtimi $\Rightarrow$ definicija z zaprtimi:

i) $A \subseteq \bigcup \operatorname{Int}(Q_j) \subseteq \bigcup Q_j$

Obrat: zaprte $\Rightarrow$ odprte.

Q kvader, npr. $Q = [a,b]$, $2Q$ je kvader, ki ga dobimo iz Q, tako da Q raztegnemo

preko središča Q raztegnemo za faktor 2.

$V(2Q) = 2^n V(Q)$

$Q \subseteq \operatorname{Int}(2Q)$

Naj bo $\varepsilon > 0$. $\exists Q_1, Q_2, \dots$ zap.:

i) $A \subseteq \bigcup_j^\infty Q_j \subseteq \bigcup_j^\infty \operatorname{Int}(2Q_j)$

ii) $\sum_j V(Q_j) < \frac{\varepsilon}{2^n}$

$$\sum V(2Q_j) = 2^n \cdot V(Q) < \varepsilon.$$

Trditev: Naj bo A kompaktna množica. Tedaj je $V(A)=0 \Leftrightarrow m(A)=0$.

Dokaz: $\Rightarrow$: že vemo

$\Leftarrow$: Naj bo $\varepsilon>0$. Obstajajo $Q_1,\dots$: $\cdot\ A \subseteq \bigcup \mathrm{Int}(Q_j)$

$\cdot\ \sum V(Q_j) < \varepsilon$

Ker je A kompaktna, imamo končno podpokritje $\Rightarrow A \subseteq \bigcup_1^N \mathrm{Int}(Q_j) \subseteq \bigcup_1^N Q_j$ in $\sum_1^N V(Q_j) \le \sum^\infty V(Q_j) < \varepsilon$. $\Big\} \Rightarrow V(A)=0$.

Če nekaj lastnost $L(x)$ velja za vse x razen za x iz množice z mero 0, pravimo, da ta lastnost velja skoraj povsod. $m(\{x;\ \text{ne velja } L(x)\})=0$

$f=g$ s.p. če je $m(\{x;\ f(x)\neq g(x)\})=0$

$\chi_{\mathbb{Q}}=0$ s.p.

$\chi_{\mathbb{Q}}$ ni zvezna v nobeni točki, ampak $g(x)=0$ je zvezna v vseh točkah.

$f(x)=\begin{cases}1; & x\ge 0\\ 0; & x<0\end{cases}$ je zvezna s.p., saj je $m(\{0\})=0$.

Ali obstaja $g:\mathbb{R}\to\mathbb{R}$ zvezna, da je $f=g$ skoraj povsod na $\mathbb{R}$.

Če ima množica mero 0, je njena notranjost prazna. $m(A)=0$ $\mathrm{Int}(A)=\emptyset$, A^c je gosta.

Če bi tak g obstajal, pak je $g(x)=1$ $\forall x>0$ in $g(x)=0$ $\forall x<0$. $\Rightarrow g$ ni zvezna v 0.

$f(x)=\begin{cases}\frac{1}{n}; & x \text{ oblike } \frac{m}{n} \text{ okrajšan}, x\in\mathbb{Q}\cap[0,1)\\ 0 & \text{sicer}\end{cases}$, $f(0)=0$

f je zvezna natanko na iracionalnih in 0 in $[0,1]$. $m(\{x\in[0,1] \mid f \text{ ni zvezna v } x\})=0$. f je zvezna skoraj povsod.

Je integrabilna, njen integral je 0.

Trditev: Naj bo $f:K\to[0,\infty)$ integrabilna, $f(x)\ge 0$ $\forall x\in\mathbb{R}$. Denimo, da je $\int_K f(x)dx=0$. Tedaj je $f(x)=0$ skoraj povsod.

Opomba: Če je f zvezna, je $f\equiv 0$ na K.

Posledica: Naj bosta f in $g:K\to[0,\infty)$ integrabilni. Če je $f(x)\le g(x)$ $\forall x$ in $\int_K f(x)dx=\int_K g(x)dx$ $\Rightarrow f=g$ skoraj povsod.

Dokaz posledice: $g-f=h$, $h\ge 0$. h je integrabilna, $\int_K h(x)dx=0$. $\Rightarrow h=0$ skoraj povsod $\Leftrightarrow f=g$ skoraj povsod.

Lema: Naj bo $f:K\to[0,\infty)$ integrabilna. Naj bo $\int_K f(x)dx=0$. Tedaj za vsak $c>0$: $V(\{x\in K \mid f(x)\ge c\})=V([f\ge c])=0$.

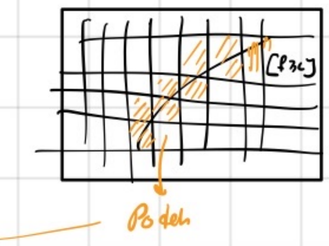
Dokaz trditve: $\{x\in K \mid f(x)>0\}=\bigcup_1^\infty \{x\in K;\ f(x)>\frac{1}{j}\}$

$V(\{x\in K \mid f(x)>\frac{1}{j}\})=0 \Rightarrow m(\{x\in K \mid f(x)>\frac{1}{j}\})=0 \Rightarrow m(\{x\in K \mid f(x)>0\})=0 \Rightarrow f=0$ s.p.

Dokaz leme: Naj bo $\varepsilon>0$ in $c>0$. Obstaja D delitev K, da je $S(f,D)<\varepsilon\cdot c$

$\int_K f(x)dx=0=\inf_D S(f,D)$

$\sum_{Q\in D} M(Q)V(Q) \ge \sum_{\substack{Q\in D\\ Q\cap[f\ge c]\neq\emptyset}} M(Q)V(Q)$



$\ge c\sum_{\substack{Q\in D\\ Q\cap[f\ge c]\neq\emptyset}} V(Q) \Rightarrow \sum_{\substack{Q\in D\\ Q\cap[f\ge c]\neq\emptyset}} V(Q)<\varepsilon$

Za $\varepsilon>0$ smo našli kvadre, ki pokrivajo $[f\ge c]$, vsota njihovih prostornin pa je pod ε.

$\Rightarrow V([f\ge c])=0$.

Opomba: Vemo že: $V(A)=0$ + $f:A\to\mathbb{R}$ omejena $\Rightarrow \exists \int_A f(x)dx=0$

$m(A)=0$ + $f:A\to\mathbb{R}$ omejena $\not\Rightarrow \int_A f(x)dx=0$

$A=[0,1]^2\cap\mathbb{Q}^2$

$m(A)=0$, $f(x)=1$. f ni integrabilna na A.

Trditev: Naj bo A omejena množica z mero 0 in $f: A \to \mathbb{R}$ integrabilna (zato omejena).
Tedaj je $\int_A f(x)dx = 0$.

Dokaz: $A \subseteq K$ kvader. $\tilde{f} = \chi_A \cdot f$. Naj bo D neka delitev K in $Q \in D$.

Poglejmo $Q \cap A^c \neq \emptyset \Rightarrow M(Q) \geq 0$ za $\tilde{f}$
$m(Q) \leq 0$ za $\tilde{f}$ ker zavzamemo 0 na Q.

To pomeni: $s(f,D) \leq 0 \leq S(f,D)$

Ker je f integrabilna: $\sup s = \inf S$, ki mora biti enak 0.

Lebesgueov izrek: Naj bo $f: K^{\text{kvad}} \subseteq \mathbb{R}^n \to \mathbb{R}$ omejena. Tedaj je f na K integrabilna $\Leftrightarrow$ f je zvezna skoraj povsod na K.

Posledica: Naj bo $A \subseteq \mathbb{R}^n$ omejena. Obstaja $V(A) \Leftrightarrow V(\partial A) = 0$.

Dokaz: $V(A)$ obstaja $\Leftrightarrow$ obstaja $\int_K \chi_A(x)dx$ $(= V(A))$. Integral $\int_A \chi_A dx$ obstaja po vsem, če je χ_A zvezna skoraj povsod na K.

Funkcija χ_A je zvezna na $\mathrm{Int}(A)$

$x \in \mathrm{Int}(A^c)$ χ_A je v okolici x enaka 0, torej $0 \Rightarrow$ tudi zvezna.

$x \in \partial A$, χ_A ni zvezna v tej okolici. $\Rightarrow$ χ_A ni zvezna na ∂A. (zavzame 0 in 1)

χ_A je zvezna skoraj povsod na $K \Leftrightarrow m(\partial A) = 0$, saj je to množica točk nezveznosti.

∂A je kompaktna množica (omejena in zap.), zato $m(\partial A) = 0 \Leftrightarrow V(\partial A) = 0$.

Posledica: Naj bo $f: A \to \mathbb{R}$ omejena (A omejena) omejena. Tedaj je f integrabilna na $A \Leftrightarrow f$ je zvezna skoraj povsod na A.
(če ima A prostornino)

Dokaz: $A \subseteq K$ kvader. f je int. na A: $\int_A f dx$ obstaja $\Leftrightarrow \int_K \tilde{f}(x)dx$ obstaja $\Leftrightarrow \tilde{f}$ je zvezna skoraj povsod na K.

$\Rightarrow$: f je integrabilna na $A \Rightarrow \tilde{f}$ je zvezna s.p. na A: $\exists E \subseteq K$, $m(E) = 0$, da je $\tilde{f}$ zv. na $K \setminus E$.

$E \cap A \subseteq A$, $m(E \cap A) = 0$. Na $A \setminus E$ se f ujema z $\tilde{f}$. V točkah $A \setminus E$ je $\tilde{f}$ zvezna $\Rightarrow f$ je zvezna v $A \setminus E$.

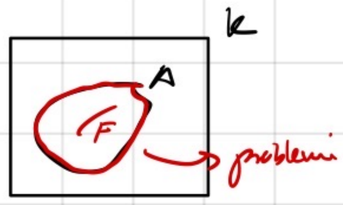
$\Rightarrow f$ je zv. skoraj povsod na A.

$\Leftarrow$: f zvezna s.p. na A. Obstaja $F \subseteq A$. $m(F) = 0$ in f je zv. na $A \setminus F$. Radi bi pokazali, da je $\tilde{f}$ zv. s.p. na K.

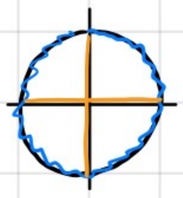
$K = \mathrm{Int}\, A \cup \partial A \cup \mathrm{Int}(A^c)$. Morebitne točke nezveznosti $\tilde{f}$ so $F \cup \partial A$. $m(F \cup \partial A) = 0$, saj $m(F) = 0$ in $V(\partial A) = 0 \Rightarrow m(\partial A) = 0$.

$\tilde{f}$ je zv. na $\mathrm{Int}(A) \setminus F$ — Tudi je zv. — $\tilde{f}$ je zv.

Torej: $\tilde{f}$ je zv. s.p. na K in zato integrabilna.



Zgled: $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ $\quad f(x,y) = \begin{cases} \sin\left(\frac{1}{xy}\right) & ; xy \neq 0 \\ 0 & \text{sicer} \end{cases}$



$V_2(A) = 0 \Rightarrow \exists V_2(A)$

f je omejena.

Nezveznosti f so vsebovane v $xy = 0$ v A. Ta množica ima mero/ploščino 0

f je torej zvezna s.p. na A. $\Rightarrow$ obstaja $\iint_A f(x,y)dxdy$

Če pa: $g(x,y) = \begin{cases} \frac{1}{xy} & xy \neq 0 \\ 0 & \text{sicer} \end{cases}$ je tudi zvezna s.p. na A (in $\mathbb{R}^2$). Ampak g ni omejena,
zato ne pride v kontekst zgoraj $\Rightarrow g$ ni običajno integrabilna na A.
Lahko pa obstaja posplošeni integral.

Dokaz izreka: $\Leftarrow$: Naj bo f zvezna s.p. na K.

Naj bo $\varepsilon > 0$. $\overset{?}{\Rightarrow} D$ delitev K: $S(f,D) - s(f,D) < \varepsilon$. $\Big\} \Rightarrow f$ je int. na K.

f je omejena, torej obstaja $M > 0$: $|f(x)| \leq M$ $\forall x \in K$. (če je $f \equiv 0$, ni težav)

$E = \{x \in K \mid f \text{ ni zvezna v } x\}$. Vemo: $m(E) = 0$, saj je f zv. s.p.

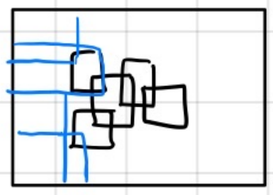
$\Rightarrow$ obstajajo $Q_1, \dots,$ kvadri, da velja: i) $E \subseteq \bigcup_j \mathrm{Int}(Q_j)$
ii) $\sum_j V(Q_j) \leq \frac{\varepsilon}{4M}$

Za $x \notin E$ je f zv. v x. Za dani ε obstaja kvader Q_x s središčem v x, da je $|f(y) - f(z)| < \frac{\varepsilon}{2V(K)}$ $\forall y,z \in Q_x$

$K \subseteq \underbrace{\bigcup_1^\infty \mathrm{Int}(Q_j)}_{\text{pokrije } E} \cup \underbrace{\bigcup_{x \in K \setminus E} \mathrm{Int}(Q_x)}_{\text{pokrije } K \setminus E}$

K je kompakt, te unije pa odprte pokrijejo K, zato obstajajo končno podpokrivanja.

$$K \subseteq \underbrace{\bigcup_{1}^{N} \mathrm{Int}(Q_j)}_{\text{prost. majhne}} \cup \underbrace{\bigcup_{i=1}^{M} \mathrm{Int}(Q_{x_i})}_{\text{velike majhne.}}$$



Po potrebi, kvadre nadomestimo s kvadri $Q_j \cap K$, $Q_{x_i} \cap K$. (odrežemo kar glede ven)

Delitev dobimo s „projekcijo":

Vsi ti kvadri porodijo delitev D kvadra K s projekcijami posameznih oglišč na robove K.

Pri taki delitvi D je vsak kvader od Q_j in Q_{x_i}; $j=1,\dots,N$, $i=1,\dots,M$, unija kvadrov delitve D.

Poglejmo: $$S(f,D)-s(f,D) = \sum_{Q\in D}(M(Q)-m(Q))V(Q) = \sum_{\substack{Q\subseteq Q_j\\ \text{za nek } j}}(M(Q)-m(Q))V(Q) + \sum_{\text{ostali}}(M(Q)-m(Q))V(Q)$$

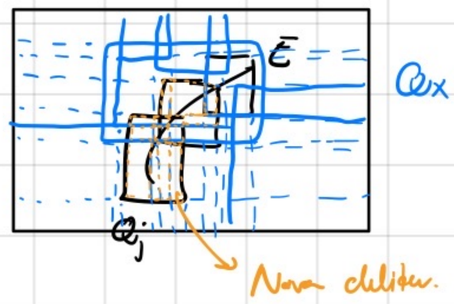
$$= \sum_{Q\subseteq Q_j}\underbrace{(M(Q)-m(Q))}_{\le 2M}V(Q) + \sum_{\text{ostali}}(M(Q)-m(Q))V(Q).$$

$$\le \sum_{Q\subseteq Q_j} 2M\cdot \underbrace{V(Q)}_{\sum V(Q)\le \frac{\varepsilon}{4M}} + \sum_{\text{ostali}} \frac{\varepsilon}{2V(K)}\cdot \underbrace{V(Q)}_{\sum V(Q)\le V(K)}$$

Dobimo: $$2M\cdot\frac{\varepsilon}{4M} + \frac{\varepsilon}{2V(K)}V(K) = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Kvader K, točka nezveznosti, pokrijemo E z Q_j, ostalo pa z Q_x.

Projiciramo robove teh množic in dobimo delitev D za K.



$\Rightarrow$: Morda na koncu semestra

Osnovne lastnosti integrala po omejenih:

Naj bo $A\subseteq\mathbb{R}^n$ omejena, $f,g: A\to\mathbb{R}$ integrabilni (tudi omejeni), $\lambda\in\mathbb{R}$.

① $f+g$, λf sta integrabilni na A in $\int_A f+g = \int_A f + \int_A g$, $\int_A \lambda f(x)dx = \lambda\int_A f(x)dx$

Množica integrabilnih funkcij na A je vektorski prostor nad $\mathbb{R}$, integral je linearen funkcional na tem prostoru.

Dokaz: To velja, ker enako velja za A = kvader, $\widetilde{(f+g)} = \tilde{f}+\tilde{g}$, $\widetilde{(\lambda f)} = \lambda\tilde{f}$

② Če je $f(x)\le g(x)$ $\forall x\in A$: $\int_A f(x)dx \le \int_A g(x)dx$.

Dokaz: Ker to velja za $A=K$ in $f(x)\le g(x) \Rightarrow \widetilde{f(x)} \le \widetilde{g(x)}$

③ Tudi $|f|$ je integrabilna na A: $\left|\int_A f(x)dx\right| \le \int_A |f(x)|dx$

Dokaz: $\widetilde{|f|} = \chi_A|f| = |\chi_A f| = |\tilde{f}|$, velja za $A=K$.

④ Če je $m\le f(x)\le M$ in A ima prostornino. Tedaj je $mV(A)\le\int_A f(x)dx\le MV(A)$.

Dokaz: Posledica 2: $g(x) = (m\cdot\chi_A)(x)$ ali $(m\cdot 1)(x)$, saj je int. ker ima prostornino.

podobno $h(x) = (M\cdot 1)(x)$, spet obstaja, saj imamo prostornino A.

Opomba: Če je $V(A)\neq 0$: $\frac{1}{V(A)}\int_A f(x)dx$, je povprečje f po A.

⑤ Naj bo A kompaktna povezana s prostornino. Naj bo $f: A\to\mathbb{R}$ zvezna. Tedaj obstaja $x_0\in A$, da je $f(x_0)V(A) = \int_A f(x)dx$

Dokaz: Če je $V(A)=0$, potem je $\int_A f(x)dx = 0$ in velja: x_0 je poljubna točka

Sicer: $f(x_0) = \frac{1}{V(A)}\int_A f(x)dx$. Vemo: $\min_A f \le \frac{1}{V(A)}\int_A f(x)dx \le \max_A f$

Ker je A kompaktna in povezana, f zavzame na A vse vrednosti med $\min_A f$ in $\max_A f$ — f zv. tudi vzamemo.

$\left(\min_A f < c < \max_A f:\ A = \underbrace{[f<c]}_{\overset{m}{\neq}\emptyset} \cup \underbrace{[f>c]}_{\overset{M}{\neq}\emptyset},\ f \text{ zvezna} \Rightarrow \text{množici sta odprti in disjunktni}\right)$

(6.) $f: A\cup B \to \mathbb{R}$ omejena. Naj bo $f: A\to\mathbb{R}$, $f: B\to\mathbb{R}$ integr. in A, B omejeni množici. $A\cup B\subseteq K$.
Naj bo $V(A\cap B)=0$. Tedaj je f int. na $A\cup B$ in $\int_{A\cup B} f(x)dx = \int_A f(x)dx + \int_B f(x)dx$

Dokaz: Spomnimo: $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ / f.

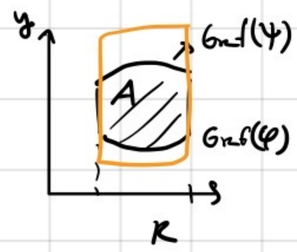
$$\chi_{A\cup B}\cdot f = \underbrace{\chi_A f}_{\text{int. na } K} + \underbrace{\chi_B f}_{\text{int. na } K} - \underbrace{\chi_{A\cap B} f}_{\text{int po trditvi: } \int_{A\cap B} f(x)=0}$$

$\Rightarrow \chi_{A\cup B} f$ je integrabilna na $K = \int_K \chi_{A\cup B} f(x)dx = \int_A f(x)dx + \int_B f(x)dx = \int_K f\chi_A + \int_K f\chi_B - \int_K f\chi_{A\cap B}$ (0)

Fubinijev izrek: $A\subseteq\mathbb{R}^n$, $B\subseteq\mathbb{R}^m$. $f: A_x\times B_y \to \mathbb{R}$ integrabilna. Če $\forall x\in A$ je $y\mapsto f^x(y)$ int na B. Tedaj

$$\iint_{A\times B} f(x,y)dxdy = \int_A\left(\int_B f^x(y)dy\right)dx$$

Trditev: Naj bo $K\subseteq\mathbb{R}^n$ kompakt in $\varphi,\psi: K\to\mathbb{R}$ zvezni in $\varphi(x)\le\psi(x)$ $\forall x\in K$.
Naj bo $\{(x,y)\in\mathbb{R}^n\times\mathbb{R} \mid \varphi(x)\le y\le\psi(x)\}$
Naj bo $f: A\to\mathbb{R}$ zvezna. Tedaj je f int. na A in



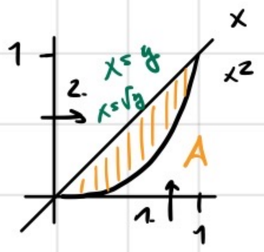
$$\iint_A f(x,y)dxdy = \int_K\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$

Dokaz: Naj bo $M>0$ tak: $|\varphi(x)|, |\psi(x)|\le M$ $\forall x\in K$. Tedaj je $A\subseteq K\times[-M,M]$
$\tilde f = \chi_A\cdot f$ integrabilna?
Točke nezveznosti $\tilde f$ so vsebovane v $\mathrm{Graf}(\varphi)\cup\mathrm{Graf}(\psi)$. Ker sta φ,ψ zv. $\Rightarrow V_{n+1}(\mathrm{Graf}(\varphi))=0$, $V_{n+1}(\mathrm{Graf}(\psi))=0$
$\Rightarrow$ Tudi $m(\mathrm{Graf}(\varphi)\cup\mathrm{Graf}(\psi))=0 \Rightarrow \tilde f$ je zvezna skoraj povsod. $\Rightarrow \tilde f$ je integrabilna na $K\times[-M,M]$.

Za $x\in K$: $y\mapsto\tilde f(x,y)$ je odsekoma zvezna funkcija ($\varphi(x), \psi(x)$ sta največ 2 točki nezveznosti) $\Rightarrow \tilde f(x,y)$ je integrabilna na $[-M,M]$.

$$\iint_A f(x,y)dxdy \overset{def}{=} \iint_{K\times[-M,M]} \tilde f(x,y)dxdy \overset{Fub}{=} \int_K\left(\int_{-M}^{M}\tilde f(x,y)dy\right)dx = \int_K\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$

Zgled: Naj bo $A\subseteq\mathbb{R}^2$ omejena s krivuljama $y=x$, $y=x^2$. Naj bo $f(x,y)=x+y$. $\iint_A (x+y)dxdy$?



$$\iint_A (x+y)dxdy = \int_0^1\left(\int_{x^2}^{x}(x+y)dy\right)dx = \int_0^1 \tfrac12 y^2 + xy\Big|_{x^2}^{x} dx = \int_0^1\left(\tfrac32 x^2 - \tfrac12 x^4 - x^3\right)dx = \tfrac12 x^3 - \tfrac14 x^4 - \tfrac{1}{10}x^5\Big|_0^1$$

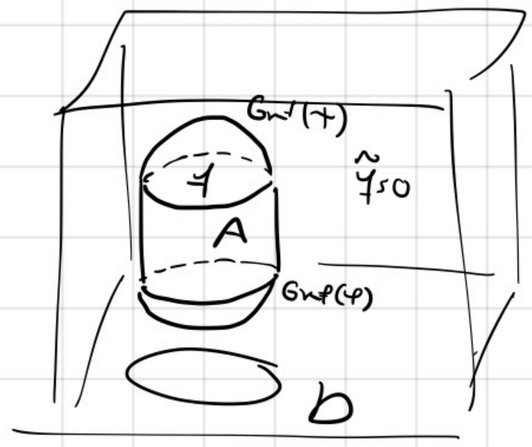
$$= \frac{10-5-2}{20} = \underline{\underline{\frac{3}{20}}}$$

Drug vrstni red: $\iint_A (x+y)dxdy = \int_0^1\left(\int_y^{\sqrt y}(x+y)dx\right)dy = \int_0^1 \tfrac12 x^2 + xy\Big|_y^{\sqrt y} dy = \int_0^1 \tfrac12 y + y^{3/2} - \tfrac12 y^2 - y^2 dy = \tfrac14 y + y^{5/2}\cdot\tfrac25 + \tfrac16 y^3 + \tfrac13 y^3\Big|_0^1$

$$= \frac14 + \frac25 - \frac12 = \frac{5+8-10}{20} = \underline{\underline{\frac{3}{20}}}$$

Izrek: Naj bo $D\subseteq\mathbb{R}^n$ s prostornino. Naj bosta $\varphi,\psi: D\to\mathbb{R}$ zvezni in omejeni, ter $\varphi(x)\le\psi(x)$ $\forall x\in D$.
Naj bo $A=\{(x,y)\in\mathbb{R}^{n+1} \mid x\in D, \varphi(x)\le y\le\psi(x)\}$. Naj bo $f: A\to\mathbb{R}$ zvezna omejena. Tedaj je f integrabilna na A in

$$\iint_A f(x,y)dxdy = \int_D\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$



Opazke in leme:
1.) D s prostornino, $\varphi: D\to\mathbb{R}$ zvezna in omejena, potem je integrabilna na D.
2.) $D\subseteq K$: $\varphi \rightsquigarrow \tilde\varphi = \chi_D\varphi$. $\tilde\varphi$ je int. na $K \Rightarrow V_{n+1}(\mathrm{Graf}(\tilde\varphi))=0$.
Tudi $\mathrm{Graf}(\varphi|_D)\subseteq\mathrm{Graf}(\tilde\varphi|_K) \Rightarrow V(\mathrm{Graf}(\varphi|_D))=0$.
3.) $B\subseteq\mathbb{R}^n$, $V_n(B)=0$. $B\times[a,b]\subseteq\mathbb{R}^{n+1}$. Potem ima $V_{n+1}(B\times[a,b])=0$.

A ima prostornino - obstaja V_{n+1} ($f=1$)

Dokaz: $D \subseteq K \subseteq \mathbb{R}^n$ (kvader) $\Rightarrow A \subseteq K \times [-M,M]$, kvader v $\mathbb{R}^{n+1}$.

$|\psi| \le M$, $|\varphi| \le M$ $\forall x \in D$.

$\tilde{f} = f \cdot \chi_A$.

Za vsak $x \in K$ je $y \mapsto \tilde{f}(x,y)$ odsekoma zvezna. $x \notin D$, je $\tilde{f}(x,y) = 0$ $\forall y$.

$$\tilde{f}(x,y) = \begin{cases} 0 & y < \varphi(x),\ y > \psi(x) \\ f(x,y) & \text{ums} \end{cases}$$, je odsekoma zvezna.

$\forall x \in K$ je $y \mapsto \tilde{f}(x,y)$ integrabilna na $[-M,M]$.

Ali je $\tilde{f}$ integrabilna na $K \times [-M,M]$?

$\Leftrightarrow$ $\tilde{f}$ zvezna s.p. na $K \times [-M,M]$. (vzbrani $\Leftarrow$).

Točke nezveznosti $\tilde{f}$ so v robu A.

Če je $\tilde{f}$ int. na $K \times [-M,M]$, potem je

$$\iint_{K\times[-M,M]} \tilde{f}(x,y)\,dx\,dy \overset{\text{Fubini}}{=} \int_K \left(\int_{-M}^{M} \tilde{f}(x,y)\,dy\right) dx \overset{x\notin D \text{ je } \tilde{f}(x,y)=0}{=} \int_D \left(\int_{-M}^{M} \tilde{f}(x,y)\,dy\right) dx = \int_D \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right) dx$$

$$\iint_A f(x,y)\,dx\,dy$$

Utemeljiti moramo še majhnost roba:

$\partial A \subseteq \mathrm{Graf}(\psi) \cup \mathrm{Graf}(\varphi) \cup (\partial D \times [-M,M])$, D ima prostornina v $\mathbb{R}^n \Rightarrow V_n(\partial D) = 0$.

i) $V_{n+1}(\mathrm{Graf}(\varphi)) = V_{n+1}(\mathrm{Graf}(\psi)) = 0$?

$\tilde{\varphi} = \varphi \chi_D$

$\tilde{\varphi}$ im nezveznosti vsebovane v $\partial D \Rightarrow \tilde{\varphi}$ je z. s.p. $\Rightarrow \tilde{\varphi}$ integrabilna na K. $\Rightarrow V_n(\mathrm{Graf}(\tilde{\varphi})) = 0$.

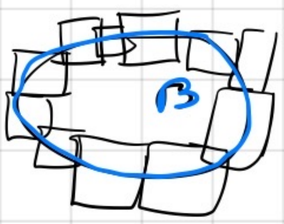
$\Rightarrow V_{n+1}(\mathrm{Graf}(\tilde{\varphi})) = 0 \Rightarrow V_{n+1}(\mathrm{Graf}(\varphi)) = 0$.

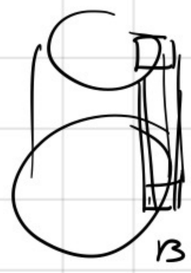
Podobno za ψ.

ii) $V_{n+1}(\partial D \times [-M,M]) = 0$? (ta je $\Rightarrow V(\partial A) = 0$).

Lema: Naj bo $B \subseteq \mathbb{R}^n$ z $V_n(B) = 0$. Naj bosta $a < b$ realna. Tedaj je $V_{n+1}(B \times [a,b]) = 0$.

Dokaz: Naj bo $\varepsilon > 0$. Obstajajo $Q_1, \dots, Q_N \subseteq \mathbb{R}^n$ kvadri, da je $B \subseteq \bigcup_1^N Q_j$ in $\sum_1^N V(Q_j) < \frac{\varepsilon}{b-a}$



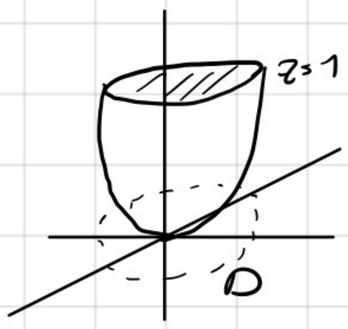


Naj bodo $K_j = Q_j \times [a,b]$, kvadri v $\mathbb{R}^{n+1}$

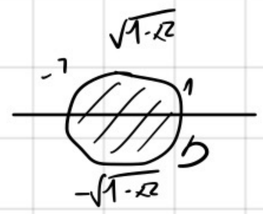
i) $B \times [a,b] \subseteq \bigcup_1^N K_j$

ii) $\sum_1^N V_{n+1}(K_j) = \sum^N V_{n+1}(Q_j)(b-a) < \varepsilon$ ■

Zgled: $T = \{(x,y,z) \in \mathbb{R}^3 \mid z \le 1,\ z \ge x^2+y^2\}$ $V(T) = ?$



$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$



$$V(T) = \iiint_T 1\,dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{1} dz\right) dx\,dy = \iint_D (1-x^2-y^2)\,dx\,dy$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\right) dx = \int_{-1}^{1} \left[2(1-x^2)\sqrt{1-x^2} - \frac{1}{3}\,2\left(\sqrt{1-x^2}\right)^3\right] dx = \frac{4}{3}\int_{-1}^{1} (1-x^2)^{3/2}\,dx = \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\,dx$$

$x^2 = t$

$t^{1/2} = x$

$dx = \frac{1}{2} t^{-1/2}\,dt$

$$= \frac{4}{3}\int_0^1 t^{\frac{1}{2}-1} (1-t)^{5/2-1}\,dt = \frac{4}{3} B\left(\frac{1}{2}, \frac{5}{2}\right) = \frac{4}{3}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)} = \frac{4}{3} \cdot \frac{\sqrt{\pi}\ \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2!} = \underline{\underline{\frac{\pi}{2}}}$$

Trditev: $\bar{K}(0,R) = \{x \in \mathbb{R}^n;\ |x| \le R\}$

$V(\bar{K}(0,R)) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^n$

$n=1$: $V_1(\bar{K}(0,R)) = \frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2})} R = \frac{\sqrt{\pi}}{\sqrt{\pi}\,\frac{1}{2}} r = 2R$

$n=2$: $V_2(\bar{K}(0,R)) = \frac{\pi}{\Gamma(2)} r^2 = \underline{\underline{\pi r^2}}$

$n=3$: $V_3(\bar{K}(0,R)) = \frac{\pi^{3/2}}{\Gamma(\frac{3}{2}+1)} r^3 = \frac{\pi^{3/2}}{\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} r^3 = \underline{\underline{\frac{4}{3}\pi r^3}}$

Dokaz: Z indukcijo: Prostornina $\bar{K}(0,R)$ obstaja. (tudi z indukcijo).

$n=1$ že vemo, da velja.

Naj trditev velja za n.

Dokazujemo, da velja za $n+1$:

$$(x_1, \dots, x_n, x_{n+1}) = (\underset{\mathbb{R}^n}{x^1}, x_{n+1}) \in \mathbb{R}^{n+1}$$

$$|x^1|^2 + |x_{n+1}|^2 \le R^2$$

$2+1$:

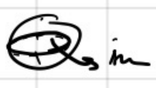
$\hookrightarrow \overline{K}(0, \sqrt{R^2 - x_{n+1}^2})$

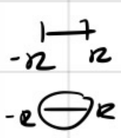
$$\iint_{\overline{K_{n+1}}(0,R)} 1\, dx\, dx_{n+1} \overset{\text{Fub.}}{=} \int_{-R}^{R} \left(\int_{K_n(0,\sqrt{R^2-x_{n+1}^2})} 1\, dx \right) dx_{n+1} = \int_{-R}^{R} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} (R^2 - x_{n+1}^2)^{n/2}\, dx_{n+1}$$

$$= \int_0^R \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} (R^2 - x_{n+1}^2)^{n/2}\, dx_{n+1} \underset{\substack{x_{n+1} = R\cdot t \\ dx_{n+1} = R\,dt}}{=} 2\int_0^1 \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^{n+1} (1-t^2)^{n/2}\, dt$$

$$= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^{n+1} \int_0^1 (1-t^2)^{n/2}\, dt \underset{\substack{t^2 = u \\ dt = \frac{1}{2} u^{1/2-1}}}{=} \frac{\cancel{2}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^{n+1} \int_0^1 \cancel{\tfrac{1}{2}} u^{1/2-1} (1-u)^{\frac{n+2}{2}-1}\, du =$$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^{n+1} \cdot B\left(\frac{1}{2}, \frac{n+2}{2}\right) = \frac{\pi^{\frac{n}{2}}}{\cancel{\Gamma(\frac{n}{2}+1)}} R^{n+1} \frac{\sqrt{\pi} \cdot \cancel{\Gamma(\frac{n}{2}+1)}}{\Gamma(\frac{n+3}{2})} = \frac{\pi^{\frac{n+1}{2}} R^{n+1}}{\Gamma(\frac{n+1}{2}+1)}$$ je ravno iskana formula.

Prostornina obstaja: $n=1$ $-R \vdash\!\dashv R$ $\quad n=3$  in prostornina, omejujem ga graf ne. funkcij.

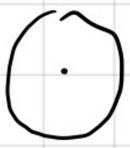
$n=2$ $-R$  R

Trditev 1: Naj bo $A \subseteq \mathbb{R}^n$ množica s prostornino. Tedaj imata prostornino $\operatorname{Int}(A)$ in $\bar{A}$ ter
$V(\operatorname{Int} A) = V(A) = V(\bar{A})$.

Trditev 2: Naj bo $A \subseteq \mathbb{R}^n$ s prostornino. Naj bo $\bar{A} \to \mathbb{R}$ omejena. Če je f integrabilna na A je integrabilna na $\bar{A}$ in $\operatorname{int} A$ in:

$$\int_A f(x)\,dx = \int_{\operatorname{Int} A} f(x)\,dx = \int_{\bar{A}} f(x)\,dx$$

Zgledi: $A \subseteq \mathbb{R}^2$, $A = K(0,R) \setminus \{0,0\}$



$f: A \to \mathbb{R}$ je omejena.

$$\iint_A f(x,y)\,dx\,dy = \iint_{K(0,R)} f(x,y)\,dx\,dy$$

Če ta obstaja, tudi desen.

$f: \bar{A} \to \mathbb{R}$

$$\iint_A f(x,y)\,dx\,dy = \iint_{K(0,R)} f(x,y)\,dx\,dy.$$

Dokaz 1: Ker ima A prostornino, je $V(\partial A) = 0$.

$\partial(\operatorname{Int} A) \subseteq \partial A \Rightarrow V(\partial(\operatorname{Int} A)) = 0$

$\partial(\bar{A}) \subseteq \partial(A) \Rightarrow V(\partial \bar{A}) = 0$

$\Rightarrow$ obstaja $V(\operatorname{Int} A)$, $V(\bar{A})$. $\quad \operatorname{Int} A \subseteq A \subseteq \bar{A} \Rightarrow V(\operatorname{Int} A) \le V(A) \le V(\bar{A})$.

$\bar{A} = \operatorname{Int} A \cup \partial A$

$\Rightarrow V(\bar{A}) = V(\operatorname{Int} A) + \underset{0}{V(\partial A)} = V(\operatorname{Int} A) \Rightarrow$ vsi so vsi enaki.

Vpeljava novih spremenljivk v n-terni integral:

Za $n=1$: $\displaystyle\int_a^b f(x)\,dx \underset{\substack{x = \varphi(t) \\ dx = \varphi'(t)\,dt}}{=} \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt$ $\qquad \varphi: [\alpha, \beta] \to [a,b]$ difeomorfizem

$\varphi: [\alpha,\beta] \to [a,b]$; $\alpha < \beta$, $a < b$.

φ je strogo monotona funkcija: i) φ je strogo naraščajoča: $\varphi(\alpha) = a$, $\varphi(\beta) = b$.: $\displaystyle\int_{[a,b]} f(x)\,dx = \int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_{[\alpha,\beta]} f(\varphi(t))\,\varphi'(t)\,dt$

ii) φ je strogo padajoča: $\varphi(\alpha)=b$, $\varphi(\beta)=a$: $\int_{[a,b]} f(x)dx = \int_a^b f(x)dx = \int_\beta^\alpha f(\varphi(t))\varphi'(t)dt = -\int_{[\alpha,\beta]} f(\varphi(t))\varphi'(t)dt$

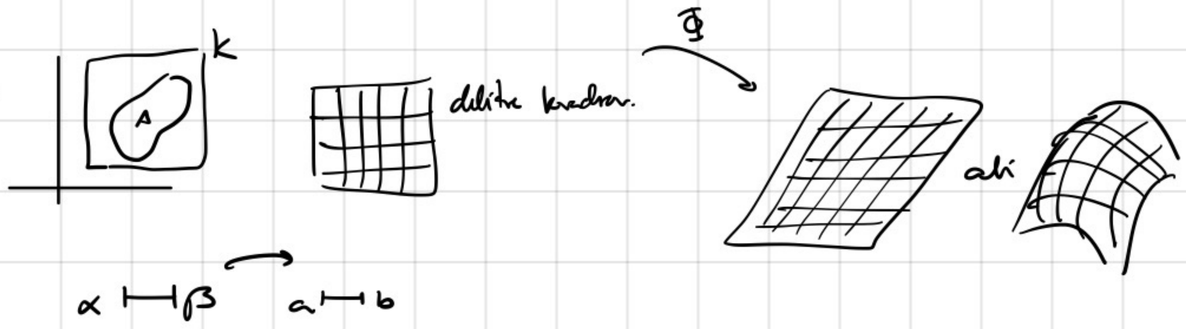
Radi bi: $\int_{[a,b]} f(x)dx = \int_{[\alpha,\beta]} f(\varphi(t))\,|\varphi'(t)|\,dt$

Izrek: Vpeljava novih spremenljivk v n-terni integral:

Naj bosta $A,B \subseteq \mathbb{R}^n$ odprti množici s prostornino (sta tudi omejeni). Naj bo $\Phi: A \to B$ C^1 difeomorfizem z omejeno Jacobijevo determinanto: $t \mapsto (J\Phi)(t)$ je omejena na A.

Naj bo $f: B \to \mathbb{R}$ integrabilna. Tedaj je $t \mapsto f(\Phi(t))\,|J\Phi(t)|$ integrabilna na A in velja $\int_B f(x)dx = \int_A f(\Phi(t))\,\overset{\det D\Phi}{|(J\Phi)(t)|}\,dt$

Opomba:



·) f int $\Leftrightarrow$ zvezna sp. na D.

·) $\det D\Phi = J\Phi \in C(A)$, omejenost, $V(A)$ obstaja $\Big\}$ $J\Phi$ je integrabilna na A

→ $f \circ \Phi = F$. Kje so potencialne točke nezveznosti?

F ni zvezna v $t \in A \Leftrightarrow f$ ni zvezna v $\Phi(t) = x$.

$E = \{x \in B \mid f \text{ ni zvezna v } x\}$ $m(E)=0$, ker f rv. sp.

$\overset{?}{\Rightarrow}$ $\Phi^{-1}(E)$

$m(\Phi^{-1}(E)) = 0$?

Trditev: $\psi: U \to V$, $U,V^{odp} \subseteq \mathbb{R}^n$, ψ C^1 difeomorfizem, $E \subseteq U$, $m(E)=0$, potem je tudi $m(\psi(E))=0$.

(Pri nas: $\psi = \Phi^{-1}: B \to A$)

Posledica: $\psi: U \to V$, $U,V^{odp} \subseteq \mathbb{R}^n$, ψ je C^1 difeomorfizem. $C \subseteq \overline{C} \subseteq U$ podmnožica s prostornino. Potem ima tudi $\psi(C)$ prostornino.

Dokaz posledice: Obstaja $V(C) \Leftrightarrow V(\partial C) = 0$ ($m(\partial C)=0$, ker kompakt). Ker je ψ difeomorfizem je $\psi(\partial C) = \partial\psi(C)$

$m(\partial\psi(C)) = m(\psi(\partial C)) \overset{\text{Trd.}}{=} 0$

$\Updownarrow$

$V(\partial\psi(C)) = 0$ oz. obstaja $V(\psi(C))$. ■

Ugotovitev: $E \subseteq \mathbb{R}^n$ ·) $m(E)=0$: ·) $\forall \varepsilon > 0$ obstaja štetno Q_j (zaprti kvadri): i) $E \subseteq \bigcup_j Q_j$

ii) $\sum_j V(Q_j) < \varepsilon$

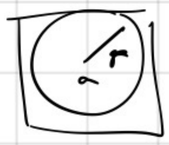
·) $\forall \varepsilon > 0$ obstaja štetno Q_j : i) $E \subseteq \bigcup_j \mathrm{Int}(Q_j)$

ii) $\sum_j V(\mathrm{Int}(Q_j)) < \varepsilon$

·) $\forall \varepsilon > 0$ obstajajo krogle K_j : i) $E \subseteq \bigcup_j K_j$

ii) $\sum_j V(K_j) < \varepsilon$

·) ⇒ ·)



$e = (1,\dots,1)$

$d_n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

$\overline{K}(a,r) \subseteq [a-re, a+re]$

Prostornine

$d_n r^n$ $\quad$ $(2r)^n$

Naj bo $\varepsilon > 0$. Iz (·): Obstajajo $K_1,\dots$ krogle: $E \subseteq \bigcup_j^\infty K(a_j,r_j) \subseteq \bigcup [a_j - r_j e, a_j + r_j e]$

in $\sum_1^\infty V(\overline{K}_j) \le \varepsilon \dfrac{d_n}{2^n} \Rightarrow \sum V([a_j - r_j e, a_j + r_j e]) < \varepsilon$

$\dfrac{2^n}{d_n} V(\overline{K}_j(a,r)) = V([a-re, a+re])$

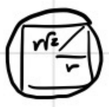
•) ⇒ ...) Obrat: razdelimo na končno manjših „kock"

$\forall \varepsilon > 0 \ \exists$ kvadri $Q_1, Q_2, \dots$: $E \subseteq \bigcup_1^\infty Q_j$

$$\sum_1^\infty V(Q_j) < \frac{\varepsilon}{d_n n^{\frac{n}{2}}}$$

Vsak Q_j razdelimo po polovici na končno mnogo kvadrov z lastnostjo, da je razmerje med najdaljšim in najkrajšim robom pod 2.

Vsakega od teh lahko povečamo do kocke: $Q_j \subseteq \tilde{Q}_j$, kjer je kocka, $V(\tilde{Q}_j) \leq 2^n V(Q_j)$



, v n-dimenzijah pa je $r = \sqrt[2]{n}$, $\boxed{\text{kocko očrtamo s kroglo}}$

Prostornina pa je $d_n r^n n^{n/2}$

Tako dobimo pokrivanje s štetno mnogo kroglami, da je $\sum_1^\infty V(K_j) < \varepsilon$

Dokaz tradicije: U^{odp} = unija štetno mnogo zaprtih kroglic,

krogle s središči v točkah z racionalnimi koordinatami z racionalnimi polmeri.

$U = \bigcup K_j$ štetna unija kroglic

$E \subseteq U \qquad E = \bigcup^\infty (K_j \cap E)$

$\psi(E) = \bigcup_1^\infty \psi(K_j \cap E)$

Dovolj je videti, da ci je $m(E) = 0 \Rightarrow m(\psi(K_j \cap E)) = 0$ (štetna unija množic z mero 0 ima mero 0).

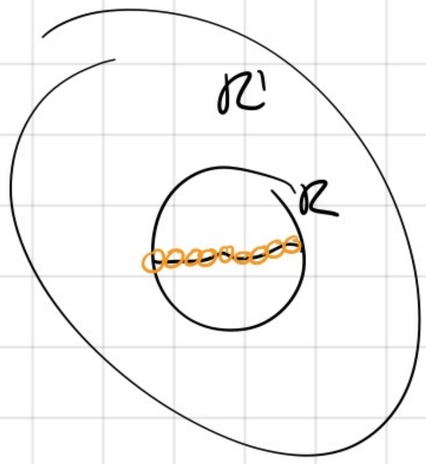
$K = \overline{K(a,R)} \subseteq U$. Naj bo $R' > R$: $\overline{K(a,R)} \subseteq \overline{K(a,R')} \subseteq U$

$E \cap \overline{K(a,R)}$

$\psi : U \to V$ C^1 difeomorfizem.

Odvod ψ se na $\overline{K(a,R')}$ omeji. $\exists c > 0$: $\forall x, x' \in \overline{K(a,R')}$:

$|\psi(x) - \psi(x')| \leq c|x - x'|$



Naj bo $B = \overline{K(\alpha,R)} \subseteq \overline{K(a,R')}$.

$\psi(B) \subseteq K(\psi(\alpha), cR)$

$V(B) = d_n r^n$

$V(\psi(B)) \leq d_n c^n r^n$

$m(E \cap \overline{K(a,R)}) = 0$

Naj bo $\varepsilon > 0$. Obstaja največ štetno $B_1, B_2, \dots$: i) $E \cap \overline{K(a,R)} \subseteq \bigcup_1^\infty B_j$ $\qquad B_j = K(\alpha_j, r_j)$

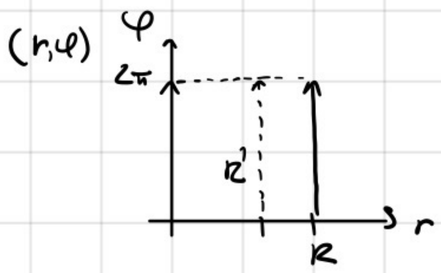
ii) $\sum_1^\infty V(B_j) < \frac{\varepsilon}{c^n}$

Iz (i) $\Rightarrow \psi(E \cap \overline{K(a,R)}) \subseteq \bigcup^\infty \psi(B_j) \subseteq \bigcup_1^\infty K(\psi(\alpha_j), cr_j)$

(ii) $\Rightarrow \sum_j^\infty V(K(\psi(\alpha_j), r_j c)) = \sum_1^\infty d_n c^n r_j^n \leq \frac{\varepsilon}{c^n} \cdot c^n = \varepsilon$. ■

$\mathbb{R}^2$: <u>polarne koordinate</u>: $x = r\cos\varphi \qquad r = \sqrt{x^2+y^2} \geq 0 \qquad \operatorname{tg}\varphi = \frac{y}{x}$

$y = r\sin\varphi \qquad \varphi \in [0, 2\pi) \qquad \varphi$ ni definiran v $(0,0)$



$\Phi : (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$

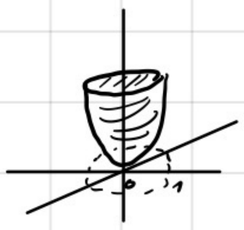
$$D\Phi(r,\varphi) = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

$J\Phi(r,\varphi) = r$ v točki 0 imamo probleme.

Če izrežemo daljico $(0,0) - (1,0)$ imamo ok difeomorfizem,

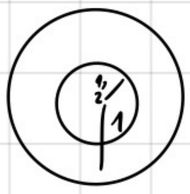
na integral to ne bo vplivalo.

Zgled: $T = \{(x,y,z) \in \mathbb{R}^3 \mid z \le 1,\ z \ge x^2+y^2\}$

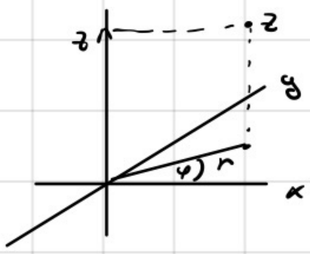


$$V(T) = \iint_{K(0,1)} (1-x^2-y^2)\,dx\,dy \overset{\text{polarne}}{=} \int_0^{2\pi}\int_0^1 \big((1-r^2)r\big)\,dr\,d\varphi = 2\pi \int_0^1 \overset{r-r^3}{(1-r^2)r}\,dr = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

Zgled: $$\iint_{K(0,1)\setminus K(0,1/2)} \frac{1}{1+x^2+y^2}\,dx\,dy \overset{\text{polarne}}{=} \int_0^{2\pi}\left(\int_{1/2}^1 \frac{r}{1+r^2}\,dr\right)d\varphi = 2\pi\cdot\frac{1}{2}\ln(1+r^2)\Big|_{1/2}^1 = \pi\left(\ln 2 - \ln\tfrac{5}{4}\right) = \pi\ln\left(\frac{8}{5}\right)$$



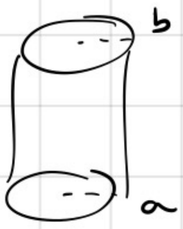
$\mathbb{R}^3$: <u>Cilindrične / Valjne koordinate:</u>

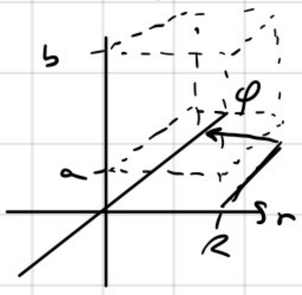


$x = r\cos\varphi$ $\quad r \ge 0$ $\quad \varphi \in [0, 2\pi)$

$y = r\sin\varphi$ $\quad r^2 = x^2+y^2$

$z = z$ $\quad \operatorname{tg}\varphi = \frac{y}{x}$





$$\Phi(r,\varphi,z) = (r\cos\varphi,\ r\sin\varphi,\ z)$$

$$D\Phi(r,\varphi,z) = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

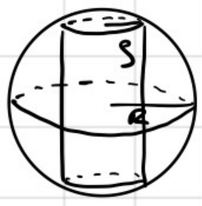
$J\Phi(r,\varphi,z) = r$

Kjer je $r=0$, je torej, ampak to ne vpliva na integral.

Zgled: $T = \{(x,y,z) \in \mathbb{R}^3 \mid z \le 1,\ z \ge x^2+y^2\}$

$$V(T) = \iiint_T dx\,dy\,dz \underset{\text{cil.}}{=} \int_0^{2\pi}\int_0^1\int_{r^2}^1 1\,r\,dz\,dr\,d\varphi$$

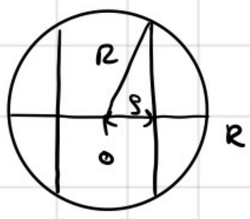
Zgled: $0 < \rho < R$



$T = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le R,\ x^2+y^2 \le \rho^2\}$

$V(T)$

Prava:



Cilindrične koordinate: $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z.$

$r^2+z^2 \le R^2 \iff |z| \le \sqrt{R^2-r^2}$ $\rightarrow$ φ nima omejitev, tečemo naravnih mejah

$r^2 \le \rho^2 \iff r \le \rho$

$$V(T) = \iiint_T 1\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\rho}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\varphi = 2\pi\int_0^{\rho} rz\Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}dr = 2\cdot 2\pi\int_0^{\rho} r\sqrt{R^2-r^2}\,dr \overset{R^2-r^2=u}{=} 4\pi\cdot\frac{2}{3}(R^2-r^2)^{3/2}\left(-\frac{1}{2}\right)\Big|_0^{\rho}$$

$$= \frac{4\pi}{3}\left(R^3-(R^2-\rho^2)^{3/2}\right) \checkmark$$

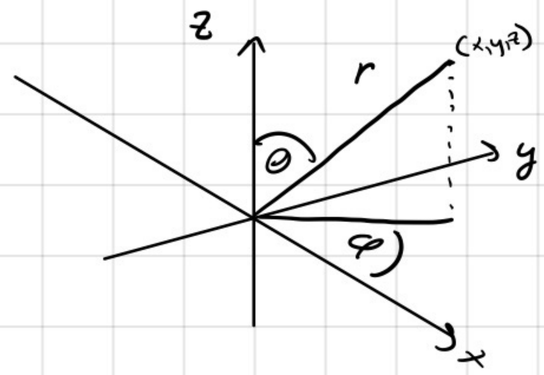
$$= \frac{4\pi R^3}{3}\left(1-\underbrace{\left(1-\left(\frac{\rho}{R}\right)^2\right)^{3/2}}\right)$$

Denimo: $\rho \ll R$: $\left(1-\left(\frac{\rho}{R}\right)^2\right)^{3/2} \approx 1-\frac{3}{2}\left(\frac{\rho}{R}\right)^2$

$\rho \ll R$: $\frac{4\pi R^3}{3}\left(\frac{3}{2}\frac{\rho^2}{R^2}\right) = 2R\pi\rho^2$ (osnovna ploskev · višina)

$\frac{4\pi}{3}(R^2-\rho^2)^{3/2}$ je zunanja dol-kor pogača pri jajčki

Sferične koordinate:



$z = r\cos\theta$ $\qquad r \geq 0,\ \varphi \in [0, 2\pi],\ \theta \in [0, \pi]$.

$x = r\sin\theta\cos\varphi$

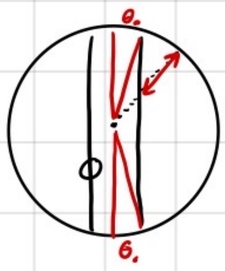
$y = r\sin\theta\sin\varphi$

$$\Phi(r,\theta,\varphi) = (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$$

$$D\Phi = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

$$\det(D\Phi) = \cos^2\theta\, r^2\sin\theta + r^2\sin^3\theta$$
$$= r^2\sin\theta\,(\cos^2 + \sin^2\theta)$$
$$= r^2\sin\theta.$$

$$\tilde{T} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq R^2,\ x^2+y^2 \geq \rho^2\}$$



Izračun $V(\tilde{T})$ v sferičnih koordinatah.

Pogoji: $\cdot)\ r^2 \leq R^2 \Rightarrow r \leq R.$

$\cdot)\ r^2\sin^2\theta \geq \rho^2$

Ni pogoja na φ, torej normalno: $\varphi \in [0, 2\pi)$

$\theta \in [\theta_0, \pi - \theta_0]$

$\Rightarrow r \geq \dfrac{\rho}{\sin\theta}$

Določimo še θ_0:

$\sin\theta_0 = \dfrac{\rho}{R}$

$\cos\theta_0 = \dfrac{\sqrt{R^2-\rho^2}}{R}$

$\operatorname{tg}\theta_0 = \dfrac{\rho}{\sqrt{R^2-\rho^2}}$ $\qquad \operatorname{ctg}\theta_0 = \dfrac{\sqrt{R^2-\rho^2}}{R}$

$$V(\tilde{T}) = \iiint_{\tilde{T}} 1\,dx\,dy\,dz = \int_0^{2\pi}\int_{\theta_0}^{\pi-\theta_0}\int_{\frac{\rho}{\sin\theta}}^{R} r^2\sin\theta\,dr\,d\theta\,d\varphi = 2\pi\int_{\theta_0}^{\pi-\theta_0} \frac{1}{3}r^3\sin\theta\Big|_{\frac{\rho}{\sin\theta}}^{R} d\theta = \frac{2\pi}{3}\int_{\theta_0}^{\pi-\theta_0} R^3\sin\theta - \frac{\rho^3}{\sin^2\theta}\,d\theta =$$

$$= \frac{2\pi}{3}\left(-R^3\cos\theta\Big|_{\theta_0}^{\pi-\theta_0} + \rho^3\operatorname{ctg}\theta\Big|_{\theta_0}^{\pi-\theta_0}\right)$$

$$= \frac{2\pi}{3}\left(2R^3\cos\theta_0 + \rho^3(-2\operatorname{ctg}\theta_0)\right)$$

$$= \frac{4\pi}{3}\left(R^3\cos\theta_0 - \rho^3\operatorname{ctg}\theta_0\right) = \frac{4\pi}{3}\left(R^3\frac{\sqrt{R^2-\rho^2}}{R} - \rho^3\frac{\sqrt{R^2-\rho^2}}{\rho}\right) = \frac{4\pi}{3}\left(\sqrt{R^2-\rho^2}\right)(R^2-\rho^2).$$ ✓

Zgled: $\vec{v_1} = (1,1,1)$

$\vec{v_2} = (2,3,1)$

$\vec{v_3} = (0,1,1)$

$$V(T) = \iiint_T 1\,dx\,dy\,dz = \|(\vec{v_1}, \vec{v_2}, \vec{v_3})\|$$
$$= \left\|\det\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right\| = \underline{\underline{2}}$$

$$\iiint_T x\,dx\,dy\,dz = ?$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A e_j = v_j$$

$A : K = [0,1]^3 \to T = A(K)$

$$A\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} u+2v \\ u+3v+w \\ u+v+w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow x = u+2v$$

$$\iiint_{T=A(K)} x\,dx\,dy\,dz = \iiint_K (u+2v)\cdot 2\,du\,dv\,dw = 2\int_0^1\int_0^1\int_0^1 (u+2v)\,du\,dv\,dw = 2\int_0^1 \tfrac{1}{2}+2v\,dw = 2\left(\tfrac{3}{2}\right) = \underline{\underline{3}}$$

Uporaba v fiziki: $\underset{\text{ploščina}}{P \subseteq \mathbb{R}^2}$, $\underset{\text{telo}}{T \subseteq \mathbb{R}^3}$

Masa: gostota: $\rho(x,y)$ za P

$\rho(x,y,z)$ za T

$$m(P) = \iint_P \rho(x,y)\,dx\,dy$$

$$m(T) = \iiint_T \rho(x,y,z)\,dx\,dy\,dz$$

Homogena plošča ali telo: $\rho(x,y,z) = \rho_0 = \text{konst.}$

Težišče: (x_P, y_P) koordinati težišča

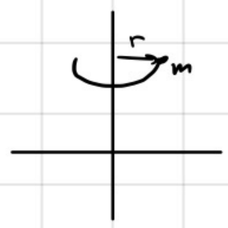
$$x_P = \frac{1}{m(P)}\iint_P x\,\rho(x,y)\,dx\,dy \qquad y_P = \frac{1}{m(P)}\iint_P y\cdot\rho(x,y)\,dx\,dy$$

Za telo: (x_T, y_T, z_T):

$$x_T = \frac{1}{m(T)}\iiint_T x\,\rho(x,y,z)\,dx\,dy\,dz \qquad y_T = \frac{1}{m(T)}\iiint_T y\,\rho(x,y,z)\,dx\,dy\,dz \qquad z_T = \frac{1}{m(T)}\iiint_T z\,\rho(x,y,z)\,dx\,dy\,dz.$$

Če je telo homogeno, se v formulah gostota ne pojavi, se pokrajša

Vztrajnostni momenti:



$J = mr^2$

T telo z gostoto $\rho(x,y,z)$: J_x, J_y, J_z

$$J_z = \iiint_T (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz$$

$$J_x = \iiint_T (y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz$$

$$J_y = \iiint_T (x^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz$$

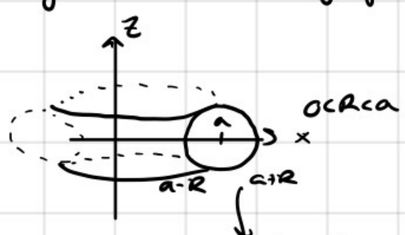
Zgled: Težišče homogene plošče , $\rho = \rho_0$

$$m = \frac{1}{2}\pi R^2\rho_0 = \iint_P \rho_0\,dx\,dy = \int_0^\pi\int_0^R \rho_0\, r\,dr\,d\varphi$$

$x_P = 0$ zaradi simetrije plošče: $\quad x_P = \frac{1}{m}\iint_P x\rho_0\,dx\,dy = \frac{1}{m}\int_0^\pi\int_0^R r\cos\varphi\, r\,dr\,d\varphi$

$$y_P = \frac{1}{m}\iint_P y\rho_0\,dx\,dy = \frac{\rho_0}{\frac{1}{2}\pi R^2\rho_0}\int_0^\pi\int_0^R r\cdot r\sin\varphi\,dr\,d\varphi = \frac{2}{\pi R^2}\int_0^\pi \frac{1}{3}R^3\sin\varphi\,d\varphi = \frac{2}{\pi}\frac{2}{3}R\cdot(-\cos\varphi)\Big|_0^\pi = \underline{\underline{\frac{4R}{3\pi}}}$$

Zgled: Vztrajnostni moment homogenega polnega torusa okoli z-osi:



$y=0$: $(x-a)^2 + z^2 = R^2$

$(r-a)^2 + z^2 = R^2$

$(\sqrt{x^2+y^2}-a)^2 + z^2 = R^2 \rightarrow$ Enačba roba torusa

$$J_z: \iiint_T \rho_0\,(x^2+y^2)\,dx\,dy\,dz \overset{\text{valjne koor.}}{=} \rho_0\int_0^{2\pi}\left(\int_{a-R}^{a+R}\left(\int_{-\sqrt{R^2-(r-a)^2}}^{\sqrt{R^2-(r-a)^2}} r^2 r\,dz\right)dr\right)d\varphi = 2\pi\rho_0\cdot 2\int_{a-R}^{a+R} r^3\sqrt{R^2-(r-a)^2}\,dr =$$

$r-a=t$

$$= 4\pi\rho_0\int_{-R}^{R}(t+a)^3\sqrt{R^2-t^2}\,dt$$

$t = R\cdot u$

$$= 4\pi\rho_0\int_{-1}^{1}(Ru+a)^3 R\sqrt{1-u^2}\,R\,du$$

$$= 4\pi R^2\rho_0\int_{-1}^{1}(\underset{0}{\underset{\shortparallel}{R^3u^3}} + \underset{\text{Cilj!!}}{3R^2u^2a} + \underset{0}{\underset{\shortparallel}{3Rua^2}} + a^3)\sqrt{1-u^2}\,du$$

$$= 4\pi R^2 \rho_0 \, 2 \cdot \int_0^1 (3R^2 u^2 a + a^3)\sqrt{1-u^2}\, du$$

$u^2 = v$
$u = v^{1/2}$
$du = \frac{1}{2} v^{\frac{1}{2}-1} du$

$$\rightarrow = 8\pi R^2 \rho_0 \int_0^1 (3R^2 a v + a^3)(1-v)^{\frac{1}{2}} \cdot \frac{1}{2} v^{\frac{1}{2}-1} du$$

$$= 4\pi \rho_0 R^2 \left(3R^2 a \int_0^1 v^{\frac{3}{2}-1} \cdot (1-v)^{\frac{3}{2}-1} du + a^3 \int_0^1 v^{\frac{1}{2}-1} (1-v)^{\frac{3}{2}-1} du \right)$$

$$= 4\pi \rho_0 R^2 \left(3R^2 a \, B\left(\tfrac{3}{2}, \tfrac{3}{2}\right) + a^3 B\left(\tfrac{1}{2}, \tfrac{3}{2}\right)\right)$$

$$= 4\pi \rho_0 R^2 \left(3R^2 a \frac{\frac{1}{2}\sqrt{\pi}\, \frac{1}{2}\sqrt{\pi}}{2} + a^3 \frac{\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{1} \right)$$

$$= 4\pi \rho_0 R^3 \left(3R^2 a \frac{\pi}{8} + a^3 \frac{\pi}{2} \right)$$

$$= 4\pi^2 \rho_0 R^2 \left(\frac{3}{8} R^2 a + \frac{a^3}{2} \right) = \rho_0 \pi R^2 \cdot 2\pi a \left(\frac{3}{4} R^2 + a^2 \right) \quad \text{oziroma:} \quad m(T)\left(\frac{3}{4}R^2 + a^2\right)$$

$$m(T) = \rho_0 \pi^2 R \cdot 2\pi a$$

Zgled: $P = \{(x,y) \in \mathbb{R}^2;\ 0 \le x \le a,\ 0 \le y \le a\}$, homogena. $\subseteq \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$

$$J_x = \iint_P (y^2 + z^2)\rho_0 \, dx\,dy = \iint_P y^2 \rho_0 \, dx\,dy = \rho_0 \int_0^a \int_0^a y^2 dy\, dx = \rho_0 a \frac{1}{3} a^3 = \frac{1}{3}\rho_0 a^2 \cdot a^2 = \frac{1}{3} m(P) \cdot a^2$$

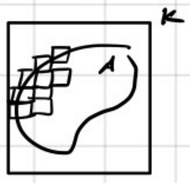
Vpeljava novih spremenljivk:

Skica dokaza izreka o vpeljavi novih spremenljivk:

Trditev: Naj bo $T: \mathbb{R}^n \to \mathbb{R}^n$ obrnljiva afina preslikava: $T(x) = Lx + d$; kjer je L obrnljiva linearna preslikava: $\det L \neq 0$. Naj bo $A \subseteq \mathbb{R}^n$ s prostornino. Tedaj je $V(T(A)) = |\det L| \cdot V(A)$.

$$\int_{T(A)} 1\, dx = \int_A |\det L| \cdot 1\, dx \qquad (L = (DT)(x),\ (JT)(x) = \det L)$$

Dokaz: $V(A)$: D delita za K



Kaj naredi translacija $x \mapsto x + d$?

Ohranja kvadre in njihovo prostornino.

Vsaka obrnljiva linearna transformacija L je produkt elementarnih transformacij. Torej je dovolj dokazati trditev za elementarne transformacije: i) $E_{ij} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & 1 & \\ & & & 1 & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix}$ i, j $\quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ za $n = 2$.

$\det(E_{ij}) = -1$, $|\det E_{ij}| = 1$. (je zamenj. i in j koordinate)

Kvadre slika v kvadre in ohranja prostornino.

Za te preslikave trditev velja.

ii) $T_i(a)$; $a \neq 0$: $\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ i $\quad \det(T_i(a)) = a$, $|\det(T_i(a))| = |a|$

Kvadre slika v kvadre, koordinata na i-tem mestu pomnoži z a.

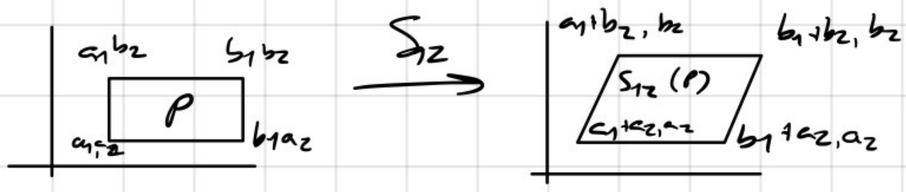
(raztegne $\vec{e}$). $V(T_i(a)(K)) = |a| \cdot V(K)$

Trditev tudi velja za te preslikave.

iii) $S_{ij} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ i, j $\quad \det(S_{ij}) = 1$

S_{ij} ne slika kvadrov v kvadre.

$n = 2$: $S_{ij} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$



Ploščini sta enaki.

K kvader, $S_{ij}(K) = P' \times K'$

P' paralelogram v ij koordinatah.

K' kvader v ostalih.

$$V(S_{ij}K) = V(K)$$

$V(\text{kvadrov delitve } D \cup A) \le V(A) \le V(\text{Kvadri, ki se dotikajo } A)$

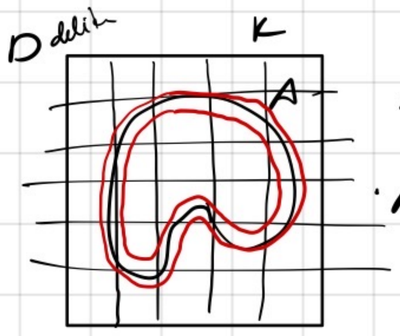
$\Rightarrow V(S_{ij}(\text{kvadrov } D, \text{ki križ. } \cup A)) \le V(S_{ij}(A)) \le V(S_{ij}(\text{Kvadri, delitve } D, \text{ki sekajo } A))$

$\Rightarrow V(A) = V(S_{ij}(A))$.

Skica dokaza izreka: $A \xrightarrow{\Phi} B$ odprti s prostornino, Φ difeomorfizem in $J\Phi$ je omejena na A.

$f: B \to \mathbb{R}$ int. (ali zv. sp.)

Trdimo: $\int_B f(x)\,dx = \int_A f(\Phi(t)) \cdot |J\Phi(t)|\,dt$, $B = \Phi(A)$



D naj bo delitev za K

$$\int_A f(\Phi(t))\,|J\Phi(t)|\,dt \doteq \sum_{Q \in D} \int_Q f(\Phi(t))\,|J\Phi(t)|\,dt$$

$$\sum_{Q \cap \partial A \neq \emptyset} \int_Q \overbrace{f(\Phi(t))\,|J\Phi(t)|}^{\text{omejeno}}\,dt < \varepsilon \text{ za dovolj fino delitev } D,$$

ker je int. sej $V(\partial A) = 0$ ($\exists V(A)$).

Dovolj je dokazovati primer, ko je A kvader, Φ je C^1 v okolici A.

Riemannova vsota

$$\int_K f(\Phi(t))\,|J\Phi(t)|\,dt \doteq \sum_{K_j \in K} f(\Phi(t_j))\,|J\Phi(t_j)| \cdot V(K_j) = \sum_{K_j \in K} f(\Phi(t_j))\,|J\Phi(t_j)|\,V(K_j - t_j) = \sum_{K_j \in K} f(\Phi(t_j)) \cdot V(D\Phi(t_j) \cdot (K_j - t_j))$$

t_j je središče K_j

translacija, ohrani prostornino

$T_x = Lx + d$

$V(T(A)) = |\det L|\, V(A)$

$$= \sum_{K_j \in K} f(\Phi(t_j))\, V(\Phi(t_j) + D\Phi(t_j)(K_j - t_j))$$

translacija

$\Phi(t_j + h) = \Phi(t_j) + D\Phi(t_j) + o(h)$

(odvedljivost)

$$V(\Phi(t_j) + D\Phi(t_j)(K_j - t_j)) \approx V(\Phi(K_j))$$

Pomožna trditev: Naj bo $\Phi: U \to V$ C^1 difeomorfizem, U, V odp. Naj bo $E \subseteq U$ kompaktna podmnožica. Naj bo $\varepsilon > 0$. Potem obstaja tak $\delta > 0$, da za vsak kvader $K_0 \in E$, katerega razmerje med najdaljšim in najkrajšim robom je manjše od 2 in vsem robovi pod δ velja:

$$|V(\Phi(K_0)) - V(D\Phi(a)K_0)| = |V(\Phi(K_0)) - V(\Phi(a) + D\Phi(a)(K_0 - a))| \le \varepsilon V(K_0) \qquad (a \text{ je središče od } K_0)$$

$$\int_K f(\Phi(t))\,|J\Phi(t)|\,dt \approx \sum_{K_j \in K} f(\Phi(t_j))\, V(\Phi(t_j) + D\Phi(t_j)(K_j - t_j))$$

$$\left| \sum_{K_j \in K} f(\Phi(t_j))\, V(\Phi(t_j) + D\Phi(t_j)(K_j - t_j)) - \sum_{K_j} f(\Phi(t_j))\, V(\Phi(K_j)) \right| \le \sum_{K_j} |f(\Phi(t_j))|\, |V(\Phi(t_j)) + D\Phi(t_j)(K_j - t_j) - V(\Phi(t_j))|$$

$$\le \sum_{K_j} |f(\Phi(t_j))| \cdot \varepsilon \cdot V(K_j) \le \sum_{K_j} V(K_j) \cdot \varepsilon \sup_B f = V(K) \cdot \varepsilon \cdot \sup_B f$$

omejen

$$\int_{\Phi(K_j)} f(\Phi(t_j))\,dx - \int_{\Phi(K_j)} f(x)\,dx \le \int_{\Phi(K_j)} |f(\Phi(t_j)) - f(x)|\,dx$$

$\le \varepsilon V(\Phi(K_j))$

enak. zv.

$$\Rightarrow \int_K f(\Phi(t))\,|J\Phi(t)|\,dt = \sum_{K_j} f(\Phi(t_j))\, V(\Phi(K_j)) = \sum_{K_j} \int_{\Phi(K_j)} f(\Phi(t_j))\,dx \overset{f \text{ zvezna}}{\approx} \sum_{K_j} \int_{\Phi(K_j)} f(x)\,dx$$

$$= \int_B f(x)\,dx, \quad B = \Phi(K) = \bigcup \Phi(K_j)$$

"■"

Posplošeni n-terni integral:

Če je vzet od funkcije ali množice neomejen, govorimo o posplošenem integralu.

Predpostavke: 1.) $A \subseteq \mathbb{R}^n$, rob A ima mero 0. $m(\partial A)=0$.

2.) $f: A \to \mathbb{R}$ je zvezna sp.

$\tilde{f} = f \cdot \chi_A = \begin{cases} f(x); & x \in A \\ 0; & x \notin A \end{cases}$, $\tilde{f}$ je zvezna sp.

Torej: $A = \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ je zvezna sp. je praktično isto potem.

f ni nujno omejena, zato: $P := \{x \in \mathbb{R}^n \mid f$ ni omejena v nobeni okolici $x\}$

- P je zaprta podmnožica $\mathbb{R}^n$: $\mathbb{R}^n - P = P^c$ je odprta podmnožica $\mathbb{R}^n$.

  P^c je odprt, saj $x_0 \in P^c \Rightarrow$ obstaja okolica, kjer je f omejena $\Rightarrow$ ta okolica je v komplementu. $\Rightarrow x_0 \in \operatorname{Int} P^c$.
- $m(P) = 0$, saj je f neznana v $x \in P$.

K_f = množica vseh kompaktnih množic s prostornino v $\mathbb{R}^n \setminus P$.

Trditev: Naj bo $Q \in K_f$. Tedaj obstaja $\int_Q f(x)dx$

Dokaz: Dovolj je pokazati, da je f omejena na Q. (ker že vemo, da $\exists V(Q)$ in f zv. sp.)

Za vsak $x \in Q$ obstaja odprta okolica U_x, da je f na $U_x \to \mathbb{R}$ omejena (x ni v P).

$Q \subseteq \bigcup_{x \in Q} U_x$, obstaja končno podpokritje $Q \subseteq U_{x_1} \cup \ldots \cup U_{x_N}$, $f: U_{x_1} \cup \ldots \cup U_{x_n} \to \mathbb{R}$ je omejena $\Rightarrow$ integral obstaja.
omejena na Q.

Naj bo $f: \mathbb{R}^n \to \mathbb{R}$ nenegativna: $f(x) \geq 0$ $\forall x$.

Potem definiramo $\int_{\mathbb{R}^n} f(x)dx = \sup_{Q \in K_f} \int_Q f(x)dx \in [0, \infty]$

Radi bi bolj enostaven način:

Definicija: Naj bo $U^{odp} \subseteq \mathbb{R}^n$ ($U = \mathbb{R}^n \setminus P$). Zaporedje kompaktnih podmnožic K_j v U izčrpa množico U, če velja:

i) $K_1 \subseteq \operatorname{Int} K_2 \subseteq K_2 \subseteq \operatorname{Int} K_3 \subseteq \ldots$

ii) $\bigcup K_i = U$.

Zgledi: 1.) $\mathbb{R}^n$: $K_j = \underbrace{\overline{K(0,j)}}_{\text{krogla}}$

2.) $\mathbb{R}^2$, $U := K(0,1) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ $\quad K_j = \overline{K(0, 1-\frac{1}{2j})}$

3.) $U = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 < 1\}$

$K_j := \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{j^2} \leq x^2+y^2 \leq (1-\frac{1}{2j})^2\}$

Za vsako odprto $U \subseteq \mathbb{R}^n$ obstaja izčrpanje s kompaktnimi množicami:

Za $U \neq \mathbb{R}^n$: $K_j := \overline{K(0,j)} \cap \{x \in U, d(x, \partial U) \geq \frac{1}{j}\}$ To je izčrpanje za U.

Obstaja izčrpanje $\mathbb{R}^n \setminus P$ s kompaktnimi podmnožicami s prostornino.

- $K_1 \subseteq \operatorname{Int} K_2 \subseteq K_2 \subseteq \operatorname{Int} K_3 \subseteq \ldots$
- $\bigcup K_i = \mathbb{R}^n \setminus P$
- K_i imajo prostornino.

Vsi $K_j \in K_f$.

$I_j = \int_{K_j} f(x)dx$ je naraščajoče zaporedje števil. $\quad \int_{K_{j+1}} f(x)dx = \int_{K_{j+1} \setminus K_j} f(x)dx + \int_{K_j} f(x)dx$

Torej obstaja $\lim_{j \to \infty} I_j$.

Trditev: $\lim_{j \to \infty} I_j = \underbrace{\lim_{j \to \infty} \int_{K_j} f(x)dx}_{I} = \underbrace{\sup_{Q \in K_f} \int_Q f(x)dx}_{J}$

Dokaz: Ker je $K_j \in K_f$ $\forall j \Rightarrow I \leq J$.

Obratna neenakost: $Q \in K_f$ $\quad Q \subseteq \mathbb{R}^n \setminus P$

$\bigcup_j K_j = \mathbb{R}^n \setminus P = \bigcup \operatorname{Int}(K_j) = \mathbb{R}^n \setminus P \Rightarrow Q \subseteq \bigcup_j \operatorname{Int} K_j \Rightarrow$ obstaja končno podpokritje $\Rightarrow$
Q kompakt

$$Q \subseteq \bigcup_{i=1}^{N} \mathrm{Int}(K_{j_i}) \subseteq K_N \Rightarrow \int_Q f(x)\,dx \le \int_{K_n} f(x)\,dx \Rightarrow \mathcal{J} \le I$$

$$\Rightarrow I = \mathcal{J}. \quad \blacksquare$$

Zgled: $f(x,y) = e^{-x^2-y^2}$, $A = \mathbb{R}^2$.

·) $K(0,j) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < j^2\}$

·) $Q_j = [-j,j]^2$.

1.) $$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{j\to\infty} \iint_{K(0,j)} e^{-x^2-y^2}\,dx\,dy \overset{\text{polarne}}{=} \lim_{y\to\infty} \int_0^{2\pi}\int_0^{j} e^{-r^2}\cdot r\,dr\,d\varphi = \lim_{y\to\infty} 2\pi\left(-\tfrac{1}{2}\right)e^{-r^2}\Big|_0^j = \pi \lim_{y\to\infty}(1-e^{-j^2}) = \pi.$$

2.) $$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{j\to\infty} \iint_{Q_j} e^{-x^2-y^2}\,dx\,dy = \lim_{j\to\infty} \int_{-j}^{j}\int_{-j}^{j} e^{-x^2-y^2}\,dy\,dx = \lim_{j\to\infty} \int_{-j}^{j} e^{-x^2}\left(\int_{-j}^{j} e^{-y^2}\,dy\right)dx = \lim_{j\to\infty}\left(\int_{-j}^{j} e^{-y^2}\,dy\right)\left(\int_{-j}^{j} e^{-x^2}\,dx\right) = \left(\int_{-\infty}^{\infty} e^{-t}\,dt\right)^2 = \pi$$

Zgled: $\displaystyle\iiint_{K(0,1)} \frac{dx\,dy\,dz}{x^2+y^2+z^2}$, $f(x) = \dfrac{1}{x^2+y^2+z^2}$, $P = \{(0,0,0)\}$. $\chi_{K(0,1)} \cdot \dfrac{1}{x^2+y^2+z^2} = \tilde{f}(x,y,z)$

Radi bi: $\displaystyle\iiint_{K(0,1)} \tilde{f}(x,y,z)\,dx\,dy\,dz$, vzirpanje pa je v $\mathbb{R}^3 \setminus \{0,0,0\}$. $K_j = \overline{K(0,j)} \setminus K(0,\tfrac{1}{j})$

$$\iiint_{K(0,1)} \tilde{f}(x,y,z)\,dx\,dy\,dz = \lim_{j\to\infty} \iiint_{K_j} \tilde{f}(x,y,z)\,dx\,dy\,dz = \lim_{j\to\infty} \iiint_{\overline{K(0,1)}\setminus K(0,\frac{1}{j})} \frac{dx\,dy\,dz}{x^2+y^2+z^2} \overset{\text{sferični}}{=} \lim_{j\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_{\frac{1}{j}}^{1} \frac{\cancel{r^2}\sin\vartheta}{\cancel{r^2}}\,dr\,d\vartheta\,d\varphi = \lim_{j\to\infty} 2\pi\cdot 2\cdot\left(1-\tfrac{1}{j}\right) = 4\pi$$

Če $f: \mathbb{R}^n \to \mathbb{R}$ predznačena: $f := f^+ - f^-$

$f^+: \mathbb{R}^n \to [0,\infty)$, $f^-: \mathbb{R} \to [0,\infty)$

$f^+ = \max\{f,0\}$, $f^- = \max\{-f,0\}$, obe ≥ 0.

Če je $f(x) \ge 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$, $f^+ - f^- = f(x)$.

$f(x) \le 0 \Rightarrow f^+(x) = 0$, $f^-(x) = -f(x)$, $f^+ - f^- = f(x)$.

Velja: $|f| = f^+ + f^-$.

Maksimum je zv. funkcija. Kadar, če je f zvezna s.p. je f^+, f^- zv. s.p.

Definicija: $f: A \to \mathbb{R}$, f zv. s.p., $m(\partial A) = 0$ je absolutno integrabilna na A, če obstaja oz. $\int_A |f(x)|\,dx$ je končen.

Ekvivalentno: če sta integrala $\int_A f^+(x)\,dx$ in $\int_A f^-(x)\,dx$ končna.

Tedaj definiramo:
$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx$$

Opomba: $$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\,dx$$

· $\displaystyle\int_0^\infty \frac{|\sin x|}{x}\,dx = \infty$, ni absolutno integrabilna

· $\displaystyle\sum^\infty a_j = \lim_{n\to\infty} S_n$, pri drugih vrstah ni nasprotnega vrstnega reda, zato absolutna konvergenca dovoli poljubno preurejanje.

Množico absolutno integrabilnih funkcij na $A \subseteq \mathbb{R}^n$ označimo z $\mathcal{L}^1(A)$.

To je vektorski prostor nad $\mathbb{R}$.

·) $f, g \in \mathcal{L}^1(A) \Rightarrow |f+g| \le |f| + |g| <$ končno + končno

·) λ, f: $|\lambda f| = |\lambda|\cdot|f|$, končno $\cdot\,\lambda$.

$f \mapsto \int_A |f|\,dx = \|f\|_1$, $f: \mathcal{L}^1(A) \to \mathbb{R}$

·) Za $f = 0$ je $\|f\|_1 = 0$.

·) $\|f\|_1 \ge 0$ $\forall f$.

·) $|\lambda|\,\|f\|_1 = \|\lambda f\|_1$

·) $\|f+g\|_1 = \int_A |f+g|\,dx \le \int_A |f|\,dx + \int_A |g|\,dx = \|f\|_1 + \|g\|_1$.

·) Če je $\|f\|_1 = 0 \Leftrightarrow \int_A |f(x)|\,dx = 0 \Rightarrow |f(x)| = 0$ s.p. $\Rightarrow f = 0$ s.p.

Da popravimo to, v $\mathcal{L}^1(A)$ vpeljemo ekvivalenčno relacijo: $f \sim g$, če je $f = g$ s.p.

1.) $f \sim f$ ✓

2.) $f \sim g \Leftrightarrow g \sim f$ trivi

3.) $f \sim g$ in $g \sim h$: $\overset{?}{\Rightarrow} f \sim h$

$$[f \neq h] \subseteq [f \neq g] \cup [g \neq h]$$

trivi $m([f \neq h]) = 0 \Rightarrow f \sim h$.

Zato vzamemo: $L^1(A) = \mathcal{L}^1(A)/_\sim$

$L^1(A)$ pa ima $\|\ \|_1$ normo definirano.

Dokaz izreka Lebesgueovega kriterija:

(K, $f: K \to \mathbb{R}$ omejena: f int. na $K \Leftrightarrow f$ zvzn sp.) $\Leftarrow$ že imamo.

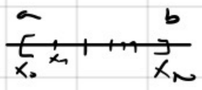
$\Leftarrow$: $E = \{x \in K \mid f$ ni zv. v $x\}$, $m(E) = 0$. Za $\varepsilon > 0$ pokrijemo E z $\operatorname{Int} Q_j$: $E \subseteq \bigcup_1^\infty Q_j$, $\sum_1^\infty V(Q_j) < \frac{\varepsilon}{4M}$, $M \geq |f|$

$x \notin E$, potem za Q_x: $y, z \in Q_x$ je $|f(y) - f(z)| < \frac{\varepsilon}{2V(K)}$

$K \subseteq \bigcup_1^N Q_j \cup Q_{x_1} \cup \ldots \cup Q_{x_m}$

$$S(D) - s(D) = \underbrace{\sum_{Q \subseteq Q_j} \underbrace{(M(Q) - m(Q))}_{\leq 2M \leq 2M \cdot \frac{\varepsilon}{4M}} \underbrace{V(Q)}_{\text{majhno}}} + \underbrace{\sum_{Q \in Q_{x_j}} \underbrace{(M(Q) - m(Q))}_{\leq \frac{\varepsilon}{2V(K)}} \underbrace{V(Q)}_{\leq V(K)}} \Rightarrow \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\Rightarrow$: Naj bo f integrabilna na K.



$n = 1$: D delitev K.

$$\sum M_j \cdot \chi_{[x_{j-1}, x_j)} = f_D^+(x)$$

sup in inf po $[x_{j-1}, x_j]$

$$\sum m_j \cdot \chi_{[x_{j-1}, x_j)} = f_D^-(x)$$

(za $n = 2$ so kvadri $\square$)

$$\int_{K = [a,b]} f_D^+(x)\,dx = S(f, D)$$

Razlika pri večih dim: $\square$ namesto $\mapsto$

$$f_D^-(x) \leq f(x) \leq f_D^+(x)$$

$$\int_K f^-(x)\,dx = s(f, D)$$

Ker je f integrabilna, so vsi Riemannovi in Darbouxovi vsoti „blizu" integrala $\int_K f(x)\,dx$, či so le dolžine delilnih intervalov dovolj majhne.

Imamo zaporedje vedno bolj finih delitev, katerih dolžine intervalov delitev gredo proti 0.

Recimo: Na vsakem koraku razpolovimo prejšnji interval

Dobimo $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$, so vedno finejše delitve

Dobimo: $f_{D_j}^-(x) \leq f(x) \leq f_{D_j}^+(x)$

Velja: $f_{D_{j+1}}^+(x) \leq f_{D_j}(x)$

$f_{D_{j+1}}^-(x) \geq f_{D_j}(x)$

Obstajata limiti $f_{D_j}^+(x) \overset{j \to \infty}{=} f^+(x)$ in $f_{D_j}^-(x) \overset{j \to \infty}{=} f^-(x)$

Torej: $f^-(x) \leq f(x) \leq f^+(x)$ $\forall x$ razen $x = b$.

$K = [a, b]$

Trdimo: f^+, f^- sta integrabilni in $\int_K f^-(x)\,dx = \int_K f(x)\,dx = \int_K f^+(x)\,dx \Rightarrow f^- = f^+$ sp.

Vemo: $\int_K f_{D_j}^+(x) = S(f, D_j) \xrightarrow{f \text{ int.}} \int_K f(x)\,dx$.

$f \leq f^+ \leq f_{D_j}^+$ $\forall j$, zato $S(f^+, D) \leq S(f_{D_j}^+, D)$

$\downarrow$ D dovolj fina

$$\int_K f_{D_j}^+(x)\,dx = S(f, D_j) \xrightarrow{j \to \infty} \int_K f(x)\,dx \Rightarrow \inf_D S(f^+, D) \leq \int_K f(x)\,dx$$

Po drugi strani: $f^+ \geq f$, zato $\Rightarrow \int_K f(x)\,dx \leq \sup_D s(f^+, D) \Rightarrow f^+$ je integrabilna in $\int_K f(x)\,dx = \int_K f^+(x)\,dx$

Podobno: $\int f^-(x)dx = \int f(x)dx \Rightarrow f^+ = f^-$ sp.

$E = \{x\in K \mid f^-(x) < f^+(x)\} \cup \{\bigcup_1^\infty D_j\}$ $\Rightarrow m(E)=0.$

$m=0$ — Šteje manjše kemčnih množic

Trdimo: Če $x\notin E \Rightarrow f$ je zv.

$x_0 \notin E$. Potem $f^+(x_0) = f^-(x_0) = f(x_0)$

Naj bo $\varepsilon>0$. $\Rightarrow$ Obstaja j_0: $f^+_{D_{j_0}}(x) - f^-_{D_{j_0}}(x_0) < \varepsilon$

$x \notin \bigcup_1^\infty D_j$ $\exists \delta>0$: $\forall x$ $|x-x_0|<\delta \Rightarrow$

$f^+_{D_{j_0}}(x) = f_{D_{j_0}}(x_0)$

$f^-_{D_{j_0}}(x) = f^-_{D_{j_0}}(x_0)$

$\Rightarrow f^+_{D_{j_0}}, f^-_{D_{j_0}}$ sta konstantni v okoli x_0. $\Rightarrow$ $f^+_{D_{j_0}}(x_0) = f^+_{D_{j_0}}(x) = \sup_{\text{okoli } x_0} f$

$f^-_{D_{j_0}}(x_0) = f^-_{D_{j_0}}(x) = \inf_{\text{okoli } x_0} f$

$\sup_{|x-x_0|<\delta} f - \inf_{|x-x_0|<\delta} f < \varepsilon$ $\Rightarrow \forall x: |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$

Pomožna trditev: $\Phi: U\to V$, C^1 difeo, $E\subseteq U$ kompaktna. Naj bo $\varepsilon>0$. Potem obstaja tak $\delta>0$, da za vsak K_0 kvader v E, katerega razmerje med najdaljšim in najkrajšim robom je pod 2 in vsemi robovi pod δ, velja, da je $|V(\Phi(K_0)) - V(D\Phi(a_0)K_0)| = |V(\Phi(K_0)) - V(\Phi(a) + D\Phi(a)(K_0-a))| < \varepsilon\cdot V(K_0)$; a_0 je središče K_0.

Dokaz: 1.) Vemo že, da $\Phi(K_0)$ ima prostornino.

2.) $J\Phi$ je omejena na E (zvezna na kompaktu je enakomerno z.), $|J\Phi|\le M$

3.) preslikava $x\mapsto D\Phi(x)$ je enakomerno zvezna

4.) $\left(\|A\| = \sup_{\|x\|=1}\|Ax\|\right)$ Funkcija $x\mapsto \|(D\Phi)(x)^{-1}\|$ je zvezna na U in enakomerno zvezna ter navdol omejena z $m>0$ na E, $m \le \|D\Phi(x)^{-1}\|$. $\forall x\in E$.

5.) BŠS: $a = \Phi(a) = 0$.

$\rho\in(0,1)$, K. $(1-\rho)K_0 \le K_0 \le (1+\rho)K_0$

$(1-\rho)D\Phi(0)K_0 \le D\Phi(0)K_0 \le (1+\rho)D\Phi(0)K_0$

$V((1+\rho)D\Phi(0)K_0) - V((1-\rho)D\Phi(0)K_0)$
$= ((1+\rho)^n - (1-\rho)^n)\,|J\Phi(0)|\,V(K_0)$

$(1+\rho)^n - (1-\rho)^n = 2\rho((1+\rho)^{n-1} + (1+\rho)^{n-2}(1-\rho) + \dots + (1-\rho)^{n-1})$
$\le 2\rho n(1+\rho)^{n-1} \le 2^n\rho n$

$|V((1+\rho)D\Phi(0)K_0) - V((1-\rho)D\Phi(0)K_0)| < 2^n\rho n M V(K_0)$. za $\rho := \frac{\varepsilon}{2^n n M}$ je $<\varepsilon$.

Če velja tudi $\underbrace{(1-\rho)D\Phi(0)K_0 \subseteq \Phi(K_0) \subseteq (1+\rho)D\Phi(0)K_0}_{\text{To rabimo.}} \Rightarrow |V(\Phi(K_0)) - V(D\Phi(0)K_0)| < \varepsilon\cdot V(K_0)$

$x = D\Phi(0)^{-1}D\Phi(0)x$

$\|x\| \le \|D\Phi(0)^{-1}\|\,\|D\Phi(0)x\| \Rightarrow m\|x\| \le \|D\Phi(0)x\|$

Če imamo daljico dolžine l, potem vsaka slika te daljice v daljico dolžine vsaj ml.

Od tod sledi: razdalja od $D\Phi(0)K_0$ do komplementa $(1+\rho)D\Phi(0)K_0$ je vsaj $m\cdot\rho\cdot\frac{1}{2}$ dolžine najkrajšega roba K_0.

$\Phi\in C^1$, zato velja da obstaja tak $\tilde\delta$ da za vsak h, $\|h\|<\tilde\delta \Rightarrow \|\Phi(a+h) - \Phi(a) - D\Phi(a)h\| \le m\cdot\rho\cdot\frac{1}{\sqrt n}\|h\|$. $\Rightarrow \le m\rho l$

Izberemo $\delta := \frac{2\tilde\delta}{\sqrt n}$

Če so vsi robovi K_0 pod $\delta \Rightarrow \forall x = a+h\in K_0$ je $|h|<\tilde\delta$ in $|h|\le 2\sqrt n\cdot$ dolžina najkrajše stranice. $= \sqrt n\, l$

$\Rightarrow |h|\le l$